

# For Generalised Algebraic Theories, Two Sorts Are Enough

Samy Avrillon  
ENS Lyon  
Lyon, France  
samy.avrillon@ens-lyon.fr

Ambrus Kaposi  
Eötvös Loránd University  
Budapest, Hungary  
akaposi@inf.elte.hu

Ambroise Lafont  
Niyousha Najmaei  
Johann Rosain  
lafont@lix.polytechnique.fr  
najmaei@lix.polytechnique.fr  
rosain@lix.polytechnique.fr  
LIX, CNRS, Inria, École polytechnique,  
Institut Polytechnique de Paris  
Palaiseau, France

## Abstract

Generalised algebraic theories (GATs) allow multiple sorts indexed over each other. For example, the theories of categories or Martin-Löf type theories form GATs. Categories have two sorts, objects and morphisms, and the latter are double-indexed over the former. Martin-Löf type theory has four sorts: contexts, substitutions, types and terms. For example, types are indexed over contexts, and terms are indexed over both contexts and types. In this paper we show that any GAT can be reduced to a GAT with only two sorts, and there is a section-retraction correspondence (formally, a strict coreflection) between models of the original and the reduced GAT. In particular, any model of the original GAT can be turned into a model of the reduced (two-sorted) GAT and back, and this roundtrip is the identity.

The reduced GAT is simpler than the original GAT in the following aspects: it does not have sort equalities; it does not have interleaved sorts and operations; if the original GAT did not have interleaved sorts and operations, then the reduced GAT won't have operations interleaved between different sorts. In a type-theoretic metatheory, the initial algebra of a GAT is called a quotient inductive-inductive type (QIIT). Our reduction provides a way to implement QIITs with sort equalities or interleaved constructors which are not allowed by Cubical Agda. An instance of our reduction is the well-known method of reducing mutual inductive types to a single indexed family. Our approach is semantic in that it does not rely on a syntactic description of GATs, but instead, on Uemura's bi-initial characterisation of the category of (finite) GATs in the 2-category of finitely complete categories with a chosen exponentiable morphism.

## CCS Concepts

• **Theory of computation** → *Type theory; Semantics and reasoning.*

## Keywords

Categorical Semantics, Inductive Types, Generalised Algebraic Theories

## 1 Introduction

*Generalised algebraic theories* (GATs), introduced by Cartmell [8], extend the notion of algebraic theories by allowing the specification of multi-sorted algebraic structures where sorts can be indexed over each other. For example, the GAT of transitive graphs has two sorts,

vertices and edges, and the sort of edges is double-indexed over the sort of objects. Syntactically, it consists of two sort declarations  $\mathbf{V} : \text{Set}, \mathbf{E} : V \rightarrow V \rightarrow \text{Set}$ , and an operation  $\mathbf{T} : \prod v_1, v_2, v_3 : V. E v_1 v_2 \rightarrow E v_2 v_3 \rightarrow E v_1 v_3$ . The category of models of this GAT is precisely the category of transitive graphs. Another example is the GAT of Martin-Löf type theory, which has four sorts: contexts, substitutions, types and terms, with various indexing between them. This GAT involves the sort declarations  $\mathbf{Con} : \text{Set}, \mathbf{Sub} : \mathbf{Con} \rightarrow \text{Set}, \mathbf{T} : \mathbf{Con} \rightarrow \text{Set}, \mathbf{Ty} : \mathbf{Con} \rightarrow \text{Set}, \mathbf{M} : \mathbf{Ty} \rightarrow \mathbf{Ty} \rightarrow \text{Set}$ .

The present work shows that any GAT can be reduced to a *family GAT*, i.e. a GAT which, essentially, starts with  $\mathbf{U} : \text{Set}, \mathbf{El} : \mathbf{U} \rightarrow \text{Set}$  and does not involve any other sort declaration. We call this reduction *two-sortification*. The idea of the translation is simple: in the original GAT, replace  $\text{Set}$  with  $\mathbf{U}$ , and insert  $\mathbf{El}$  in front of any used sort; finally prepend the result with  $\mathbf{U} : \text{Set}, \mathbf{El} : \mathbf{U} \rightarrow \text{Set}$ . For example, applying the translation to the GAT of transitive graphs yields  $\mathbf{U} : \text{Set}, \mathbf{El} : \mathbf{U} \rightarrow \text{Set}, \mathbf{V} : \mathbf{U}, \mathbf{E} : \mathbf{El} V \rightarrow \mathbf{El} V \rightarrow \mathbf{U}, \mathbf{T} : \prod v_1, v_2, v_3 : \mathbf{El} V. \mathbf{El}(E v_1 v_2) \rightarrow \mathbf{El}(E v_2 v_3) \rightarrow \mathbf{El}(E v_1 v_3)$ .

## 1.1 Motivation

This reduction has concrete applications in the implementation of inductive types in type theory. Different classes of inductive types correspond to different classes of algebraic theories, e.g. mutual inductive types are initial models of equation-free multi-sorted algebraic theories; quotient-inductive-inductive types (QIITs) are initial models of (a variant of) Cartmell's GATs, and similarly, inductive-inductive types (IITs) are initial models of equation-free GATs. Reducing between different classes of GATs is important for implementation. For instance, IITs are not supported by Rocq, but can be recovered via their reduction to indexed inductive types [15]. Two-sortification reduces general QIITs to the smaller class of two-sorted QIITs, which is useful on its own: for example, Sestini [26] defines another reduction of IITs to indexed inductive types, but only considers two-sorted IITs for simplicity. His simplification is justified by the current paper. Additionally, two-sortification simplifies the GAT and its corresponding QIIT in the following three respects.

1. *Eliminating Sort Equations.* Two-sortification gets rid of equations between expressions of type  $\text{Set}$ , which we call *sort equations*. As an example, consider the extension of the GAT of type theory with a Russell universe, that is, with a type constructor<sup>1</sup>

<sup>1</sup>Additional equations should ensure that  $R$  is compatible with substitution, and indexing is needed to avoid Russell's paradox.

$R : \prod \Gamma, \text{Ty } \Gamma$  and a sort equation  $\text{Tm } \Gamma (R \Gamma) =_{\text{Set}} \text{Ty } \Gamma$ , which expresses that terms of type  $R$  are identified with types. In the reduced GAT, this equation becomes  $\text{Tm } \Gamma (R \Gamma) =_U \text{Ty } \Gamma$ , which is no longer a sort equation.

Therefore, two-sortification provides a workaround for Cubical Agda's lack of support for sort equations in QITs. This technique has been used, for example, by Altenkirch and Scoccola [3] to define the integers as a pointed type with a sort equation.

**2. Interleaved Operators.** Agda does not support defining inductive types with interleaved operators for different sorts. An example is the syntax of Martin-Löf type theory with sorts **Con**, **Sub**, **Ty**, **Tm**. The functor law  $A[\gamma \circ \delta] = A[\gamma][\delta]$  is a constructor of **Ty**, but it refers to the substitution composition constructor of **Sub**, while the constructor for substitution extension  $-, - : (\gamma : \text{Sub}(\Delta, \Gamma)) \times \text{Tm}(\Delta, A[\gamma]) \rightarrow \text{Sub}(\Delta, \Gamma.A)$  refers to the type substitution constructor  $-[-]$  in **Ty**. Consequently, two-sortification has been used in practice, for example by Altenkirch et al. [2], to implement the syntax of type theory in Cubical Agda.

**3. Interleaved Sorts and Operators.** In the theory of System  $F_\omega$ , types are indexed by kinds and terms are indexed by types at kind  $*$ ; thus, the operator  $*$  has to be declared before the sort of terms. The two-sortified version of System  $F_\omega$  has only two sorts **U** and **El**. As a result, two-sortification transforms interleaved sorts and operators into interleaved operators of different sorts. In point 2 above, we explained how two-sortification removes interleaved operators; hence, by applying two-sortification once more, we can also remove interleaved operators of different sorts.

## 1.2 What is a GAT?

A GAT can be thought of as a context in a logical framework or domain-specific type theory: this has been the guiding intuition in the examples given so far. Substitutions between such contexts are the morphisms. It has occasionally been argued [14, 19] that GATs are inconvenient to work with directly because of their low-level definition as raw syntax and typing judgements [8]. Here, we adopt a more high-level approach by leveraging a simple universal property of the category of finite GATs due to Uemura [27]. Concretely, the category of finite GATs, equipped with the projection  $p_G : (\mathbf{U} : \text{Set}, \mathbf{u} : \mathbf{U}) \rightarrow (\mathbf{U} : \text{Set})$ , is bi-initial among finitely-complete categories equipped with an exponentiable morphism. Exponentiability means that the pullback functor along this morphism has a right adjoint, called the pushforward along that morphism. Bi-initiality means that given any cartesian category  $\mathbb{C}$  with a chosen exponentiable morphism  $p'$ , there exists a unique (up to unique isomorphism) finite-limit-preserving functor  $F$  from the category of GATs to  $\mathbb{C}$  such that  $F(p_G) \cong p'$  and pushforwards along  $p_G$  are mapped (up to isomorphism) to pushforwards along  $p'$ .

Type-theoretic perspectives on (variants of) GATs can be found in the literature [5, 14, 20]. These provide (stricter) initiality properties and avoid dealing with untyped syntax. To construct a functor from the category of GATs to some category  $\mathbb{C}$ , this approach requires upgrading  $\mathbb{C}$  with the structure of a category with families [9] with suitable type formers. In contrast, the above bi-initiality characterisation requires elementary categorical structures on  $\mathbb{C}$ : finite

limits and a chosen exponentiable morphism, which the functor is expected to map the GAT morphism  $(\mathbf{U} : \text{Set}, \mathbf{u} : \mathbf{U}) \rightarrow (\mathbf{U} : \text{Set})$  to.

As a preliminary example, we can easily reconstruct the *model functor* to (large) categories that computes the category of models of a given GAT: since we expect the exponentiable GAT morphism  $(\mathbf{U} : \text{Set}, \mathbf{u} : \mathbf{U}) \rightarrow (\mathbf{U} : \text{Set})$  to be mapped to the forgetful functor from the category of pointed sets to the category of sets, we simply choose  $p'$  to be this (exponentiable) functor. We will see that two-sortification can be defined in a similar a way, by equipping the category of GATs sliced over the GAT of families with a suitable exponentiable morphism.

## 1.3 Contributions

**A Syntactic and Semantic Account of Two-sortification.** We give a formal account of two-sortification as an endofunctor on the category of GATs. Moreover, for any theory  $\Gamma$ , we construct a GAT morphism from its translation  $T\Gamma$  to  $\Gamma$ , which we call the *coreflector morphism* of  $\Gamma$ , as well as a GAT morphism from  $T\Gamma$  to the theory  $\mathbf{U} : \text{Set}, \mathbf{El} : \mathbf{U} \rightarrow \text{Set}$ , intuitively forgetting everything from  $T\Gamma$  but the two sorts. The mapping  $\Gamma \mapsto (T\Gamma \rightarrow (\mathbf{U}, \mathbf{El}))$  actually induces a fully faithful functor to the slice category over the theory  $\mathbf{U}, \mathbf{El}$ .

On the semantic side, we define the *model functor*, from the category of GATs to the category of locally small categories, that computes the category of models of a given GAT. We show that the image of the coreflector morphism by the model functor has a left adjoint inducing a strict coreflection between the categories of models of the original GAT and its two-sortification: the roundtrip starting from a model of the original GAT is the identity.

**Example 1.1.** Consider the GAT of transitive graphs. The right adjoint maps a model  $(\mathbf{U}, \mathbf{El}, V, E, T)$  of the translated GAT to a transitive graph whose underlying set of vertices is  $\mathbf{El}(V)$ , whose set of edges between two vertices  $a$  and  $b$  is  $\mathbf{El}(E(a, b))$ , and whose transitive operation is given by  $T$ . Conversely, the left adjoint maps a transitive graph  $G = (V, E, T)$  to the family defined by  $\mathbf{U}_G = \{*\} + V \times V$ ,  $\mathbf{El}_G(*) = V$ , and  $\mathbf{El}_G(a, b) = E(a, b)$ . The transitive operation is again given by  $T$ . The roundtrip starting from a transitive graph yields the same transitive graph.

We also provide a simple description of the category of models of the reduced GAT that we now sketch. First, let us define the *family functor* of a GAT as the composition of the coreflective left adjoint and the projection mapping a model of the reduced GAT to its underlying (set-indexed) family. Let  $\mathbf{El}_M : \mathbf{U}_M \rightarrow \mathbf{Set}$  denote the image of a model  $M$  of the original GAT by the family functor. We show that the category of models of the reduced GAT is, up to isomorphism, the category defined as follows:

- An object consists of a model  $M$  of the original GAT, a family  $(\mathbf{U}', \mathbf{El}')$ , and a function  $f : \mathbf{U}_M \rightarrow \mathbf{U}'$  such that  $\mathbf{El}'(f(u)) = \mathbf{El}_M(u)$  for each  $u \in \mathbf{U}_M$ ;
- A morphism between two objects consists of a morphism between the underlying models and a morphism between the underlying families compatible with the underlying functions.

In other words, denoting by  $F$  the family functor, the category of models of the reduced GAT is, up to isomorphism, the full subcategory of the comma category  $F/\mathbf{Fam}$  spanned by morphisms that are in the canonical (splitting) cleavage of the fibration from families to sets.

**Example 1.2.** By definition, the family functor of the GAT of transitive graphs maps a transitive graph  $G = (V, E, T)$  to the family  $(U_G, El_G)$  defined as in Example 1.1. Instantiating the above description of the category of models of the reduced GAT, an object consists of a transitive graph  $G = (V, E, T)$ , a family  $(U', El')$ , and a function  $f: U_G \rightarrow U'$  such that  $El'(f(u)) = El_G(u)$  for each  $u \in U_G$ . Giving such a function  $f$  amounts to picking an element  $V'$  in  $U'$ , and providing an operation  $E': V \times V \rightarrow U'$ . In this way, the equation reformulates as  $El'(V') = V$  and  $El'(E'(a, b)) = E(a, b)$ . This means that  $V$  and  $E$  are uniquely determined from  $(U', El', V', E')$ . Finally, the transitive operation  $T$  has exactly the right type to make  $(U', El', V', E', T)$  a model of the translated GAT. This induces a bijection between objects of this category and the models of the translated GAT, that extends to an isomorphism of categories.

*Universal property of infinite GATs.* In the type-theoretic perspective, infinite GATs can be thought of as infinite contexts, such as, for example, the signature of semi-simplicial types [21]. We show that infinite GATs enjoy a bi-initial characterisation similar to that of finite GATs: we merely have to consider small limits instead of finite limits. Although we mainly work with finite GATs for simplicity, it is straightforward to check that all our results actually extend to infinite GATs thanks to this characterisation.

*Initiality models of GATs.* We prove that every GAT has an initial model, exploiting the above mentioned bi-initial characterisation.

## 1.4 Related Work

*Formal definition of GATs.* As hinted above, we heavily rely on Uemura’s bi-initial characterisation of GATs [27]. Garner [10] provides an alternative categorical definition of GATs, as algebras for a certain monad on a category of presheaves.

We also already mentioned related work with a type-theoretic perspective on GATs, studying variants of Cartmell’s original notion. Kovács’s thesis [19] develops the theory of finitary QIITs with sort equations (Chapter 4, see also [14]), and the theory of infinitary QIITs without sort equations (Chapter 5, see also Kovács and Kaposi [20]). Infinitary QIITs allow operations with infinitely many arguments, a feature absent from GATs. These works feature signatures that can be infinite but in some restricted way, e.g., the GAT of semi-simplicial types above mentioned is out of scope. Finally, let us mention that we follow their convention regarding the direction of the morphisms of GATs, which differs from Cartmell (in particular, a morphism of GAT induces a functor between the categories of models in the same direction).

*Initial Models of GATs.* In the spirit of Initial Algebra Semantics [11], an initial model of a GAT can be thought of as the categorical account of the corresponding inductive type with its full dependent recursion principle (see, e.g., [14, Section 7.3]).

The above cited works about signatures for QIITs provide proofs of existence of initial models for their respective notions of GATs.

Our own proof is actually inspired by them. Cartmell’s original paper on GATs [8] states (without proof) that the image of any GAT morphism by the model functor has a left adjoint, thus providing an initial model of any GAT by considering GAT morphisms to the empty theory. This is also a consequence of Cartmell’s translation from GATs to essentially algebraic theories (although the translation of sort equalities is not much detailed).

There is also a line of research consisting in constructing initial models by reducing some notion of GATs to a simpler one. In this respect, two-sortification can be understood as an extension of the reduction of mutual inductive types to indexed inductive types [16]. As explained above, it eliminates some features of GATs such as sort equations. The strict coreflection ensures the existence of the initial model of the original GAT whenever the reduced one has an initial model. In fact, Sestini [26] explicitly conjectured the validity of two-sortification for his variant of GATs to justify focusing on the two-sorted ones, before reducing them further to indexed-inductive types, in a type-metatheoretic setting. Nonetheless, our proof of initiality is direct and does not rely on the reduction to two-sorted GATs.

## 1.5 Synopsis

In Section 2, we recall some preliminaries. Importantly, we review Uemura’s bi-initial characterisation of finite GATs. In Section 3, we define two-sortification as a bi-initial functor from finite GATs to GATs sliced over the theory  $\mathbf{Fam}$  of families. In Section 4, we construct the category of models of the translated GAT from the category of models of the original GAT, from which we deduce the coreflection between the two categories of models. In Section 5, we prove that each finite GAT has an initial model. The arguments developed there are used in Section 6 to show that the two-sortification functor is fully faithful. Finally, in Section 7, we discuss how to extend our results to infinite GATs, by extending the bi-initiality property of the category of finite GATs.

For the reader’s convenience, we provide hyperlinks from occurrences of a notion to its definition.

A few definitions and results of this paper have been formalised in Rocq. We provide this formalisation as supplementary material [4]. A detailed account of what has been formalised can be found in the README file.

## 2 Preliminaries

This section recalls various definitions and results that we use throughout the paper, most importantly the bi-initial characterisation of GATs in Section 2.2.

### 2.1 Exponentiable morphisms

In this subsection, we recall the definition of exponentiable morphisms in a category with finite limits, and various basic properties and definitions related to them.

**Definition 2.1** ([27, Definition 2.4]). A *cartesian category* is a category with finite limits. A morphism  $f: Y \rightarrow X$  in a cartesian category  $\mathbb{C}$  is said to be *exponentiable* if the pullback functor  $f^*: \mathbb{C}/X \rightarrow \mathbb{C}/Y$  along  $f$  has a right adjoint, where  $\mathbb{C}/Z$  denotes the slice category over an object  $Z$ : objects are morphisms to  $Z$  and

morphisms between them are commuting triangles. We call this right adjoint the *pushforward* along  $f$  and denote it with  $f_*$ .

**Proposition 2.2** ([25, Corollary 1.2]). *A morphism  $f : Y \rightarrow X$  is exponentiable in the sense of Definition 2.1 if and only if the following composite has a right adjoint, where  $\text{dom} : \mathbb{C}/Y \rightarrow \mathbb{C}$  denotes the functor mapping a morphism to  $Y$  to its domain.*

$$\mathbb{C}/X \xrightarrow{f^*} \mathbb{C}/Y \xrightarrow{\text{dom}} \mathbb{C}$$

**Notation 2.3.** We denote the right adjoint of the composite in Proposition 2.2 by  $P_f : \mathbb{C} \rightarrow \mathbb{C}/X$ . We usually conflate  $P_f$  with  $P_f \circ \text{dom} : \mathbb{C} \rightarrow \mathbb{C}$ , as it will be clear from the context which one is meant.

**Definition 2.4.** Let  $\mathbb{C}$  and  $\mathbb{C}'$  be categories with finite limits, with  $\mathbb{C}$  equipped with an exponentiable morphisms  $f : Y \rightarrow X$ . Let  $F : \mathbb{C} \rightarrow \mathbb{C}'$  be a functor preserving pullbacks and such that  $Ff$  is exponentiable. We say that  $F$  preserves pushforwards *along*  $f$  if the canonical natural transformation below right, as the mate [18, §2.2] of the isomorphism below left, is also an isomorphism.

$$\begin{array}{ccc} \mathbb{C}/Y & \xleftarrow{f^*} & \mathbb{C}/X \\ \downarrow F & \cong & \downarrow F \\ \mathbb{C}'/FY & \xleftarrow{(Ff)^*} & \mathbb{C}'/FX \end{array} \quad \begin{array}{ccc} \mathbb{C}/Y & \xrightarrow{f_*} & \mathbb{C}/X \\ \downarrow F & \cong & \downarrow F \\ \mathbb{C}'/FY & \xrightarrow{(Ff)_*} & \mathbb{C}'/FX \end{array}$$

**Lemma 2.5** ([27, Proposition 2.12]). *In the context of Definition 2.4, the functor  $F : \mathbb{C} \rightarrow \mathbb{C}'$  preserves pushforwards along  $p$  if and only if it commutes with the associated right adjoints  $P_p$  and  $P_{Fp}$  (up to isomorphism), i.e. the canonical natural transformation from  $F \circ P_p$  to  $P_{Fp} \circ F$  is an isomorphism.*

## 2.2 Generalised Algebraic Theories

In this section, we define GATs and introduce the universal properties of the category of finite GATs that we use throughout the paper.

**Definition 2.6.** The category **FinGat** of *finite generalised algebraic theories* is the category of contexts and substitutions of the type theory generated by a universe **Set** of types, extensional equality types, and dependent products over types in **Set**.

Equality types provide, for each pair of terms  $(t, u)$  of the same type  $A$ , a new type  $t = u$  with the usual expected rules. Dependent product over types in **Set** means that if  $\Gamma \vdash A : U$  and  $B$  is a type in the context  $\Gamma, x : A$ , we can form the product type  $\Pi_{x:A} B$  in context  $\Gamma$  and we also have lambda-abstraction, application, as well as the usual  $\beta$  and  $\eta$  rules. A substitution from  $\Gamma$  to  $x_1 : A_1, \dots, x_n : A_n$  is a list of terms  $t_1, \dots, t_n$  such that  $\Gamma \vdash t_i : A_i[x_j \mapsto t_j]_{1 \leq j < i}$ .

We will not detail further this definition, as our main technical tool is the following bi-initial characterisation of GATs.

**Theorem 2.7** ([27, Theorem 4.1]). *Let  $p_{\mathcal{G}} : Y_{\mathcal{G}} \rightarrow X_{\mathcal{G}}$  be the projection morphism  $(A : \mathbf{Set}, a : A) \rightarrow (A : \mathbf{Set})$  in **FinGat**. We sometimes drop the index  $\mathcal{G}$  when no confusion arises.*

*The category **FinGat**, equipped with  $p_{\mathcal{G}}$ , is bi-initial in the 2-category **CartExp** defined as follows:*

- an object is a **CartExp**-category  $\mathbb{C}$ , that is, a cartesian category  $\mathbb{C}$  equipped with an exponentiable morphism  $p : Y \rightarrow X$  in  $\mathbb{C}$ ;

- a morphism, which we call a **CartExp**-functor, from  $(\mathbb{C}, p)$  to  $(\mathbb{C}', p')$  is a finite-limit-preserving functor  $F : \mathbb{C} \rightarrow \mathbb{C}'$  equipped with an isomorphism  $F(p) \cong p'$ , preserving pushforwards along  $p$  in the sense of Definition 2.4;
- a 2-cell between two morphisms is a natural transformation between the underlying functors.

Bi-initiality of **FinGat** as stated by Theorem 2.7 means that given any object  $(\mathbb{C}, p)$  in **CartExp**, there exists a **CartExp**-functor **FinGat**  $\rightarrow \mathbb{C}$ , and moreover, any pair of such functors are naturally isomorphic, for a unique isomorphism.

**Definition 2.8.** We call any **CartExp**-functor **FinGat**  $\rightarrow (\mathbb{C}, p)$  a *bi-initial (CartExp)-functor*, and sometimes even calls it *the* bi-initial **CartExp** functor to  $\mathbb{C}$  although it is only unique up to isomorphism.

We explain later in this section what the pushforward along the exponential morphism  $p_{\mathcal{G}}$  is. For now, let us note that we can always strictify a bi-initial functor.

**Proposition 2.9.** *We say that a **CartExp**-functor  $F : (\mathbb{C}, Y \xrightarrow{p} X) \rightarrow (\mathbb{C}', Y' \xrightarrow{p'} X')$  is strict if the isomorphism  $F(p) \cong p'$  is an identity. Given any **CartExp**-category  $\mathbb{C}$  with exponential arrow  $p : Y \rightarrow X$  such that  $X \neq Y$ , any **CartExp** morphism from  $\mathbb{C}$  is isomorphic to a strict one. In particular, there exists a strict morphism from **FinGat** to any **CartExp**-category.*

As a first example of application of bi-initiality, we recover the functor that computes the category of models of a given GAT.

**Example 2.10.** Consider the category **CAT** of locally small categories, equipped with the (exponentiable) projection **PtdSet**  $\rightarrow \mathbf{Set}$  from pointed sets to sets. The pullback along that functor maps a functor  $F : \mathbb{D} \rightarrow \mathbf{Set}$  to its category of elements  $\int F$ . The right adjoint maps a category  $\mathbb{D}$  to the *family fibration* **Fam**( $\mathbb{D}$ )  $\rightarrow \mathbf{Set}$ , where **Fam**( $\mathbb{D}$ ) is the category of (set-indexed) families of  $\mathbb{D}$ : an object is a pair of a set  $A$  and a family  $(d_a)_{a \in A}$  of objects of  $\mathbb{D}$  indexed by  $A$  [13, Definition 1.2.1], which we can equivalently see as a functor from  $A$  (seen as a discrete category) to  $\mathbb{D}$ . A morphism from  $F : A \rightarrow \mathbb{D}$  to  $G : B \rightarrow \mathbb{D}$  consists of a function  $H : A \rightarrow B$  and a family of  $\mathbb{D}$ -morphisms  $(h_a : Fa \rightarrow GHa)_{a \in \text{ob } A}$ .

**Notation 2.11.** We sometimes denote **Fam**(**Set**) by **Fam**: this is the category of families  $(El(A))_{A \in U}$  of sets indexed by some set  $U$ .

**Definition 2.12.** By Proposition 2.9, we get a strict **CartExp**-functor from **FinGat** to **CAT** which we call the *model functor*. We denote the image of a theory  $\Gamma$  (resp. GAT morphism  $\sigma$ ) by  $\llbracket \Gamma \rrbracket$  (resp.  $\llbracket \sigma \rrbracket$ ). We say that a category  $\mathbb{C}$  is *specified* by a theory  $\Gamma$  if  $\mathbb{C}$  is isomorphic to  $\llbracket \Gamma \rrbracket$ .

We are now in better position to understand what the right adjoint of the pullback along  $p_{\mathcal{G}}$  is, as formally described in [27, Proposition 3.34]. The guiding intuition is that  $P\Gamma$  specifies **Fam**( $\llbracket \Gamma \rrbracket$ ), since the model functor preserves pushforwards along  $p_{\mathcal{G}}$ . We provide a few examples in Table 1.

We end this section with a useful result for constructing or showing equality of natural transformations between functors from **FinGat**, used throughout the paper.

**Theorem 2.13.** *Let  $\mathbb{C}$  be a **CartExp**-category with chosen exponentiable morphism  $p_{\mathbb{C}} : Y_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ . Let  $F : \mathbf{FinGat} \rightarrow \mathbb{C}$  be a*

**Table 1: Examples of GATs and their images by  $P$** 

$\Gamma$	$P\Gamma$ specifying families of $\llbracket \Gamma \rrbracket$
$\mathbf{B} : \text{Set}$	$\mathbf{A} : \text{Set}, \mathbf{B} : \mathbf{A} \rightarrow \text{Set}$
$\mathbf{B} : \text{Set}, \mathbf{b} : B$	$\mathbf{A} : \text{Set},$ $\mathbf{B} : \mathbf{A} \rightarrow \text{Set}, \mathbf{b} : \prod a : \mathbf{A}. B a$
(Transitive graphs) $\mathbf{V} : \text{Set},$ $\mathbf{E} : \mathbf{V} \rightarrow \mathbf{V} \rightarrow \text{Set},$ $\mathbf{T} : \prod v_1, v_2, v_3 : \mathbf{V},$ $E v_1 v_2 \rightarrow E v_2 v_3 \rightarrow E v_1 v_3$	$\mathbf{A} : \text{Set}$ $\mathbf{V} : \mathbf{A} \rightarrow \text{Set}$ $\mathbf{E} : \prod a : \mathbf{A}. \mathbf{V} a \rightarrow \mathbf{V} a \rightarrow \text{Set}$ $\mathbf{T} : \prod a : \mathbf{A}. \prod v_1, v_2, v_3 : \mathbf{V} a,$ $E a v_1 v_2 \rightarrow E a v_2 v_3 \rightarrow E a v_1 v_3$

functor preserving pullbacks along<sup>2</sup>  $p_{\mathcal{G}}$ , and  $G : \mathbf{FinGat} \rightarrow \mathbb{C}$  be a **CartExp**-functor. Given a pullback square in  $\mathbb{C}$  as follows:

$$\begin{array}{ccc} FY_{\mathcal{G}} & \xrightarrow{x} & GY_{\mathcal{G}} \\ Fp_{\mathcal{G}} \downarrow & \lrcorner & \downarrow Gp_{\mathcal{G}} \\ FX_{\mathcal{G}} & \xrightarrow{y} & GX_{\mathcal{G}}, \end{array}$$

there exists a unique natural transformation  $\alpha_{(x,y)}$  between  $F$  and  $G$ :

$$\begin{array}{ccc} & F & \\ \text{FinGat} & \xrightarrow{\alpha_{(x,y)}} & \mathbb{C} \\ & G & \end{array}$$

such that the given pullback is the naturality square for  $\alpha_{(x,y)}$  at  $p_{\mathcal{G}}$ .

**PROOF.** A pullback square as above induces a **CartExp**-structure on the comma category  $F/\mathbb{C}$ , see Section A.2. The natural transformation  $\alpha_{(x,y)}$  is then given by the initial **CartExp**-functor to  $F/\mathbb{C}$ , exploiting the universal property of the comma category [7, Proposition 1.6.3].  $\square$

### 3 Syntactic Translation

In this section, we define the two-sortification functor  $T$  as a (bi-initial) **CartExp**-functor mapping a GAT to its translation equipped with its projection to the GAT of families.

The construction is given in Section 3.1. In Section 3.2, we show that the translated GAT is indeed a *family GAT*, i.e. a GAT whose only sorts are (essentially)  $\mathbf{U} : \text{Set}$  and  $\mathbf{El} : \mathbf{U} \rightarrow \text{Set}$ . Finally, in Section 3.3, for each theory  $\Gamma$ , we construct a GAT morphism  $\text{dom}(T\Gamma) \rightarrow \Gamma$ , which we call the *coreflector morphism* of  $\Gamma$ : we will show later that its image by the model functor is the right adjoint of the coreflection between  $\llbracket \Gamma \rrbracket$  and  $\llbracket T\Gamma \rrbracket$ .

#### 3.1 Two-sortification by Bi-Initiality

The goal of this section is to define the two-sortification functor.

**Notation 3.1.** Overloading Notation 2.11, we denote the theory  $PX_{\mathcal{G}} \cong (\mathbf{U} : \text{Set}, \mathbf{El} : \mathbf{U} \rightarrow \text{Set})$  of families by **Fam**.

We define two-sortification as a **CartExp**-functor from **FinGat** to the slice category **FinGat/Fam**, whose objects are GATs with a morphism to **Fam**, and morphisms are morphisms between the

<sup>2</sup>In practice, we will always apply this result with functors  $F$  preserving any pullback.

underlying GATs compatible with the morphisms to **Fam**. To that end, by Theorem 2.7, we merely need to equip **FinGat/Fam** with a suitable **CartExp**-structure. We start with a general fact regarding limits in slice categories.

**Proposition 3.2.** ***FinGat/Fam** is a cartesian category: the identity morphism on **Fam** is terminal in **FinGat/Fam**, and pullbacks are computed as in **FinGat**.*

We now need to choose an exponentiable morphism in the slice category **FinGat/Fam**. The choice is dictated by what we expect the two-sortification of  $p_{\mathcal{G}} : (\mathbf{A} : \text{Set}, \mathbf{a} : \mathbf{A}) \rightarrow (\mathbf{A} : \text{Set})$  to be: the morphism  $(\mathbf{U} : \text{Set}, \mathbf{El} : \mathbf{U} \rightarrow \text{Set}, \mathbf{A} : \mathbf{U}, \mathbf{a} : \mathbf{El} \mathbf{A}) \rightarrow (\mathbf{U} : \text{Set}, \mathbf{El} : \mathbf{U} \rightarrow \text{Set}, \mathbf{A} : \mathbf{U})$ . Let us introduce some notation to simplify the reading of such GATs.

**Notation 3.3.** We denote a generalised algebraic theory  $(\mathbf{U} : \text{Set}, \mathbf{El} : \mathbf{U} \rightarrow \text{Set}, \Gamma)$  over  $(\mathbf{U} : \text{Set}, \mathbf{El} : \mathbf{U} \rightarrow \text{Set})$  as  $\Gamma_{\text{Fam}}$ , leaving the projection to **Fam** implicit. For example, we sometimes denote **Fam** by  $(\ )_{\text{Fam}}$  and the theory  $(\mathbf{U} : \text{Set}, \mathbf{El} : \mathbf{U} \rightarrow \text{Set}, \mathbf{A} : \mathbf{U})$  by  $(\mathbf{A} : \mathbf{U})_{\text{Fam}}$ .

To complete the **CartExp**-structure on **FinGat/Fam**, we need to prove that the morphism  $(\mathbf{A} : \mathbf{U}, \mathbf{a} : \mathbf{El} \mathbf{A})_{\text{Fam}} \rightarrow (\mathbf{A} : \mathbf{U})_{\text{Fam}}$  is exponentiable. For that matter, it will be useful to re-construct it using the available operations in a **CartExp**-category (pullbacks and pushforwards along the exponentiable morphism), since then we can rely on standard results about exponentiable morphisms.

**Proposition 3.4.** *Let  $\mathbb{C}$  be **CartExp**-category with exponentiable morphism  $p : Y \rightarrow X$ . We define  $\underline{p} : \underline{Y} \rightarrow \underline{X}$  by the below right pullback, where*

- $\underline{X}$  is defined as the below left pullback;
- $P$  is the right adjoint to the pullback functor  $p^*$  along  $p$ ;
- $\epsilon : \underline{X} \rightarrow X$  is the counit of the adjunction  $p^* \dashv P$ .

$$\begin{array}{ccc} \underline{X} & \xrightarrow{\quad} & Y \\ \downarrow & \lrcorner & \downarrow p \\ PX & \xrightarrow{\quad} & X \end{array} \quad \begin{array}{ccc} \underline{Y} & \xrightarrow{\quad} & Y \\ \downarrow p^! & \lrcorner & \downarrow p \\ \underline{X} & \xrightarrow{\epsilon} & X \end{array} \quad (1)$$

Taking  $(\mathbb{C}, p) = (\mathbf{FinGat}, p_{\mathcal{G}})$ , we recover:

- $(\mathbf{A} : \mathbf{U})_{\text{Fam}} \rightarrow (\ )_{\text{Fam}}$  as the morphism  $\underline{X} \rightarrow PX$  on the left;
- $(\mathbf{A} : \mathbf{U}, \mathbf{a} : \mathbf{El} \mathbf{A})_{\text{Fam}} \rightarrow (\mathbf{A} : \mathbf{U})_{\text{Fam}}$  as  $\underline{p}$ .

**PROOF.**  $\underline{X}$  is defined as the pullback of  $Y = (\mathbf{A} : \text{Set}, \mathbf{a} : \mathbf{A})$  and  $PX = (\mathbf{U} : \text{Set}, \mathbf{El} : \mathbf{U} \rightarrow \text{Set})$  over  $(\mathbf{A} : \text{Set})$ , so it is  $(\mathbf{U} : \text{Set}, \mathbf{El} : \mathbf{U} \rightarrow \text{Set}, \mathbf{A} : \mathbf{U})$ . The claimed result follows from the counit  $\underline{X} \rightarrow (\mathbf{A} : \text{Set})$  selecting the term  $\mathbf{El} \mathbf{A}$  in the context  $\underline{X}$ .  $\square$

With this characterisation, it is straightforward to check that the morphism  $\underline{p} : \underline{Y} \rightarrow \underline{X}$  from Proposition 3.4 is exponentiable in  $\mathbb{C}/PX$ .

**Proposition 3.5.** *In the setting of Proposition 3.4, the morphism  $\underline{p} : \underline{Y} \rightarrow \underline{X}$  is exponentiable in  $\mathbb{C}/PX$ , as a morphism between  $\underline{X} \rightarrow PX$  defined in Eq. (1) and  $\underline{Y} \xrightarrow{\underline{p}} \underline{X} \rightarrow PX$ .*

**PROOF SKETCH.** Exponentiable morphisms are stable under pullbacks, and if a morphism  $f : Y \rightarrow X$  in a category  $\mathbb{C}$  with finite

limits is exponentiable, then for any morphism  $X \rightarrow Z$ , the morphism  $f$  is also exponentiable in the slice category  $\mathbb{C}/Z$ .  $\square$

From this we get two-sortification by initiality.

**Definition 3.6.** By Proposition 2.9 and Proposition 3.5, there exists a strict initial morphism from  $\mathbf{FinGat}$  to  $\mathbf{FinGat}/\mathbf{Fam}$  mapping  $p_{\mathcal{G}}$  to  $(A : U, a : El A)_{\mathbf{Fam}} \rightarrow (A : U)_{\mathbf{Fam}}$ . We call it the *two-sortification functor*  $T$ . We often conflate  $T\Gamma$  with its domain, seen as a GAT over  $\mathbf{Fam}$ .

### 3.2 Soundness

In this section, we show that the image of any GAT by the two-sortification functor is (up to isomorphism) a family GAT.

**Definition 3.7.** A *family GAT* is a finite GAT starting with  $U : \mathbf{Set}$  and  $El : U \rightarrow \mathbf{Set}$ , which does not involve any other sorts. Let  $\mathbf{FamGat}$  denotes the full subcategory of  $\mathbf{FinGat}/\mathbf{Fam}$  spanned by family GATs.

The category of family GATs inherits the  $\mathbf{CartExp}$ -structure of  $\mathbf{FinGat}/\mathbf{Fam}$  given by Proposition 3.5 in the following sense.

**Proposition 3.8.** *The category  $\mathbf{FamGat}$  of family GATs has finite limits, includes the morphism  $(A : U, a : El A)_{\mathbf{Fam}} \rightarrow (A : U)_{\mathbf{Fam}}$  which is also exponentiable in  $\mathbf{FamGat}$ . Moreover, the embedding  $\mathbf{FamGat} \hookrightarrow \mathbf{FinGat}/\mathbf{Fam}$  preserves finite limits as well as pushforwards along this exponentiable morphism.*

**Corollary 3.9.** The image of any GAT by the two-sortification functor is isomorphic to a family GAT.

**PROOF.** By uniqueness of the bi-initial  $\mathbf{CartExp}$ -functor, the two-sortification functor is isomorphic to the composition of the bi-initial  $\mathbf{CartExp}$ -functor from  $\mathbf{FinGat}$  to  $\mathbf{FamGat}$  with the embedding  $\mathbf{FamGat} \hookrightarrow \mathbf{FinGat}/\mathbf{Fam}$ . This composition maps any GAT to a family GAT by definition.  $\square$

Finally, note that if we define two-sortification as the strict bi-initial  $\mathbf{CartExp}$ -functor from  $\mathbf{FinGat}$  to  $\mathbf{FamGat}$ , then the two-sortification of any GAT is a family GAT on the nose, rather than up to isomorphism. Adopting this definition does not change anything in the rest of this paper since we only rely on the fact that two-sortification is a (strict) bi-initial  $\mathbf{CartExp}$ -functor to  $\mathbf{FinGat}/\mathbf{Fam}$ .

### 3.3 The Coreflector Morphism of a GAT

In this section, we define, for each theory  $\Gamma$ , a morphism from its two-sortification  $T\Gamma$  to  $\Gamma$ . We call it the *coreflector morphism* of  $\Gamma$ . The naming is motivated by the fact that, in Section 4, we show that the image of this morphism by the model functor is the right adjoint of the strict coreflection between the categories of models of  $\Gamma$  and  $T\Gamma$ .

**Definition 3.10.** By Theorem 2.13, the (right) pullback square defining  $p$  in Equation (1) uniquely extends to a natural transformation as below:

$$\begin{array}{ccc}
 & \mathbf{FinGat}/\mathbf{Fam} & \\
 T \nearrow & \downarrow & \searrow \text{dom} \\
 \mathbf{FinGat} & \xlongequal{\quad} & \mathbf{FinGat}
 \end{array}$$

The component of this natural transformation at a theory  $\Gamma$  is a GAT morphism  $R_{\Gamma} : T\Gamma \rightarrow \Gamma$  which we call the *coreflector morphism* of  $\Gamma$ .

**Example 3.11.** By definition, the bottom and top horizontal morphisms in the right pullback square (1) for  $\mathbb{C} = \mathbf{FinGat}$  are the coreflector morphisms of  $(A : \mathbf{Set})$  and  $(A : \mathbf{Set}, a : A)$  respectively. More explicitly, the coreflector morphism of  $(A : \mathbf{Set})$  is the substitution  $(A : U)_{\mathbf{Fam}} \rightarrow (A : \mathbf{Set})$  induced by the term  $(A : U)_{\mathbf{Fam}} \vdash El A : \mathbf{Set}$ . Its image by the model functor maps  $(U, El, A)$  to the set  $El(A)$ . As claimed in the introduction, it has a coreflective left adjoint mapping a set  $X$  to the model defined by  $U := \{*\}$ ,  $El * := X$ , and  $A := *$ . If we apply this left adjoint to a set  $X$  and then the right adjoint, we indeed get back  $X$ .

Similarly, the image by the model functor of the coreflector morphism of  $(A : \mathbf{Set}, a : A)$  is a functor from  $[(A : U, a : El A)_{\mathbf{Fam}}]$  to  $[(A : \mathbf{Set}, a : A)]$  mapping a model  $(U, El, A, a)$  to the pointed set  $(El(A), a)$ . Again, it has a coreflective left adjoint mapping a pointed set  $(X, x)$  to the model defined by  $U := \{*\}$ ,  $El * := X$ ,  $A := *$ , and  $a := x$ .

## 4 Semantic Translation

Consider the image  $[[T\Gamma]] \rightarrow [[\Gamma]]$  of the coreflector morphism of a theory  $\Gamma$  by the model functor. The goal of this section is to construct a left adjoint such that the unit of the adjunction is an identity: this is the claimed strict coreflection. In Section 4.2, we provide an explicit description of the models of  $T\Gamma$  in terms of the models of  $\Gamma$ , from which we deduce a strict coreflection between  $[[T\Gamma]]$  and  $[[\Gamma]]$ . Section 4.3 shows that the right adjoint is indeed the image of the coreflector morphism of  $\Gamma$  by the model functor.

We start with a preliminary section to define the *family functor* of a theory  $\Gamma$ , a key ingredient in the explicit description of the models of  $T\Gamma$ .

### 4.1 The Family Functor of a GAT

Intuitively, the family functor of a theory  $\Gamma$  is the composition of the coreflective left adjoint  $[[\Gamma]] \rightarrow [[T\Gamma]]$  with the projection  $[[T\Gamma]] \rightarrow [\mathbf{Fam}] \cong \mathbf{Fam}(\mathbf{Set})$ .

**Example 4.1.** In Example 3.11, we described the coreflective left adjoints for the theories  $(A : \mathbf{Set})$  and  $(A : \mathbf{Set}, a : A)$ . It follows that the family functor of  $(A : \mathbf{Set})$  maps a set  $X$  to the family  $(U, El)$  defined by  $U = \{*\}$  and  $El * = X$ . Precomposing it with the forgetful functor from pointed sets to sets yields the family functor of  $(A : \mathbf{Set}, a : A)$ .

Although this definition of the family functor via the coreflection is helpful to gain intuition, we cannot rely on it in the general case, since we have not yet defined the coreflection for an arbitrary theory  $\Gamma$ . Instead, we once again exploit bi-initiality of  $\mathbf{FinGat}$ . More specifically, we define a suitable  $\mathbf{CartExp}$ -category whose objects are functors to  $\mathbf{Fam}(\mathbf{Set})$ , so that the bi-initial  $\mathbf{CartExp}$ -functor to that category maps a theory  $\Gamma$  to its category of models  $[[\Gamma]]$  equipped with its family functor. The main subtlety lies in choosing the right notion of morphisms. It is tempting to consider commuting triangles: a morphism between  $\mathbb{C} \xrightarrow{F} \mathbf{Fam}$  and  $\mathbb{C}' \xrightarrow{F'} \mathbf{Fam}$  would be a functor  $G : \mathbb{C} \rightarrow \mathbb{C}'$  such that  $F = F' \circ G$ , so that we get the slice category  $\mathbf{CAT}/\mathbf{Fam}$ . Unfortunately, with this notion of morphism,

the bi-initial **CartExp**-functor would not meet our expectations. In particular, because it preserves the terminal objects, it would map the empty theory to  $\mathbf{Fam} \xrightarrow{\text{id}} \mathbf{Fam}$ , instead of the family functor  $\llbracket () \rrbracket = 1 \rightarrow \mathbf{Fam}$  of the empty theory. But by suitably relaxing the notion of morphism as follows, we recover the right terminal objects, and more generally the expected (finite) limits.

**Proposition 4.2.** *Let  $\text{CAT} // \mathbf{Fam}$  denote the colax slice category of locally small categories over  $\mathbf{Fam}$ : objects are functors to  $\mathbf{Fam}$  and a morphism between  $\mathbb{C} \xrightarrow{F} \mathbf{Fam}$  and  $\mathbb{C}' \xrightarrow{F'} \mathbf{Fam}$  is a functor  $G : \mathbb{C} \rightarrow \mathbb{C}'$  and a natural transformation between  $F' \circ G$  and  $F$ . Composition is defined in the obvious way.*

*Then,  $\text{CAT} // \mathbf{Fam}$  has limits and the functor  $\text{CAT} // \mathbf{Fam} \rightarrow \text{CAT}$  preserves them.*

**PROOF.** This is a direct application of [12, Corollary 4.3], noting that  $\text{CAT} // \mathbf{Fam} \rightarrow \text{CAT}$  is a bifibration.  $\square$

It remains to choose an exponentiable morphism in  $\text{CAT} // \mathbf{Fam}$ . For that matter, we expect the bi-initial **CartExp**-functor  $\text{FinGat} \rightarrow \text{CAT} // \mathbf{Fam}$  to map  $p_{\mathcal{G}} : (A : \text{Set}, a : A) \rightarrow (A : \text{Set})$  to the forgetful functor  $\llbracket p_{\mathcal{G}} \rrbracket$  from pointed sets to sets, with a suitable natural transformation involving the family functors. Based on Example 4.1, the identity natural transformation is an obvious candidate.

**Theorem 4.3.** *The forgetful functor from pointed sets to sets, equipped with the identity natural transformation between their family functors from Example 4.1, is exponentiable in  $\text{CAT} // \mathbf{Fam}$ , and the functor  $\text{CAT} // \mathbf{Fam} \rightarrow \text{CAT}$  preserves pushforwards along it.*

*Therefore, combining with Proposition 4.2,  $\text{CAT} // \mathbf{Fam}$  is a **CartExp**-category and the functor  $\text{CAT} // \mathbf{Fam} \rightarrow \text{CAT}$  is a **CartExp**-functor.*

**PROOF.** A detailed proof is given in Section A.3.  $\square$

**Corollary 4.4.** *There is a strict bi-initial **CartExp**-functor from  $\text{FinGat}$  to  $\text{CAT} // \mathbf{Fam}$  that maps a theory  $\Gamma$  to its category of models  $\llbracket \Gamma \rrbracket$  equipped with a functor to  $\mathbf{Fam}$ , which we call the *family functor* of  $\Gamma$ .*

**PROOF.** By uniqueness of the initial morphism, the composition  $\text{FinGat} \rightarrow \text{CAT} // \mathbf{Fam} \rightarrow \text{CAT}$  is isomorphic to the model functor. Moreover, because the projection  $\text{CAT} // \mathbf{Fam} \rightarrow \text{CAT}$  is an isofibration, we can refine the **CartExp**-functor  $\text{FinGat} \rightarrow \text{CAT} // \mathbf{Fam}$  so that it coincides on the nose with the model functor when composed with the projection to  $\text{CAT}$ .  $\square$

## 4.2 Description of the Models of the Reduced GAT

In this section, we give an explicit description of the models of  $T\Gamma$  in terms of the models of  $\Gamma$  and its family functor, defined in Section 4.1. More specifically, this section is devoted to the proof of the following statement.

**Theorem 4.5.** *Given a theory  $\Gamma$  with family functor  $F_{\Gamma} : \llbracket \Gamma \rrbracket \rightarrow \mathbf{Fam}$ , the category of models of its two-sortification is isomorphic to the full subcategory  $F_{\Gamma}/_c \mathbf{Fam}$  of the comma category  $F_{\Gamma}/\mathbf{Fam}$  spanned by cartesian morphisms. Explicitly,*

- *an object is a pair of a model  $M$  of  $\Gamma$  and a morphism of families  $f : F_{\Gamma}(M) \rightarrow (U, El)$  in  $\mathbf{Fam}$  that is cartesian in the sense that, for every  $u \in U'$ , the function  $f_u : El'(u) \rightarrow El(f_U(u))$  is the identity function<sup>3</sup>, where  $El' : U' \rightarrow \mathbf{Set}$  is the family  $F_{\Gamma}(M')$ ;*
- *a morphism from  $F_{\Gamma}(M) \rightarrow (U, El)$  to  $F_{\Gamma}(M') \rightarrow (U', El')$  consist of a morphism of models  $M \rightarrow M'$  and a morphism of families  $(U, El) \rightarrow (U', El')$  in  $\mathbf{Fam}$  making the obvious square commute.*

*This isomorphism is compatible with the projections of  $\llbracket T\Gamma \rrbracket$  and  $F_{\Gamma}/_c \mathbf{Fam}$  to  $\mathbf{Fam}$ , where the latter maps a cartesian morphism  $F_{\Gamma}(M) \rightarrow (U, El)$  to  $(U, El)$ , in the sense that the following triangle commutes up to isomorphism and moreover, this isomorphism is natural in  $\Gamma$ .*

$$\begin{array}{ccc} \llbracket T\Gamma \rrbracket & \xrightarrow{\cong} & F_{\Gamma}/_c \mathbf{Fam} \\ & \searrow & \swarrow \\ & \mathbf{Fam} & \end{array}$$

We already illustrated an instance of this theorem for the case of the GAT of transitive graphs in Examples 1.1 and 1.2. As an immediate consequence, we get a coreflection between the categories of models of a GAT and its two-sortification.

**Corollary 4.6.** *Given a theory  $\Gamma$  with family functor  $F_{\Gamma} : \llbracket \Gamma \rrbracket \rightarrow \mathbf{Fam}$ , there is a strict coreflection between  $\llbracket \Gamma \rrbracket$  and  $\llbracket T\Gamma \rrbracket$ : through the isomorphism  $\llbracket T\Gamma \rrbracket \cong F_{\Gamma}/_c \mathbf{Fam}$ , the right adjoint maps a cartesian morphism  $F_{\Gamma}(M) \rightarrow (U, El)$  to  $M$ , and the left adjoint maps a model  $M$  to the identity morphism  $F_{\Gamma}(M) \rightarrow F_{\Gamma}(M)$ . It follows that the family functor  $F_{\Gamma}$  is, up to isomorphism, the composite of the left adjoint and the projection  $\llbracket T\Gamma \rrbracket \rightarrow \mathbf{Fam}$ .*

This coreflection is in fact determined by the coreflector morphism of  $\Gamma$  defined in Section 3.3: the right adjoint is indeed recovered as the image of this morphism by the model functor. We defer the proof of this fact to Section 4.3.

In the rest of this subsection, we tackle the proof of Theorem 4.5. The core argument consists in upgrading the below mappings into suitable **CartExp**-functors from  $\text{FinGat}$  to  $\text{CAT}/\mathbf{Fam}$ , seen as a **CartExp**-category by Proposition 3.5.

$$\Gamma \mapsto (\llbracket T\Gamma \rrbracket \rightarrow \mathbf{Fam}) \quad \Gamma \mapsto (F_{\Gamma}/_c \mathbf{Fam} \rightarrow \mathbf{Fam})$$

Then, by bi-initiality, those two functors are isomorphic, yielding the desired natural isomorphism between  $\llbracket T\Gamma \rrbracket$  and  $F_{\Gamma}/_c \mathbf{Fam}$ .

Let us focus on extending the first mapping. The key insight is that  $\Gamma \mapsto (\llbracket T\Gamma \rrbracket \rightarrow \mathbf{Fam})$  can be recovered as the composition of  $T$  with a suitable **CartExp**-functor from  $\text{FinGat}/\mathbf{Fam}$  to  $\text{CAT}/\mathbf{Fam}$ , thanks to the following proposition.

**Proposition 4.7.** *The mapping  $(\mathbb{C}, Y \xrightarrow{p} X) \mapsto (\mathbb{C}/PX, \underline{Y} \xrightarrow{\underline{p}} \underline{X})$  described in Proposition 3.5 extends to an endofunctor  $-/\mathbf{Fam}$  on **CartExp**. Given a **CartExp**-functor  $F : (\mathbb{C}, p) \xrightarrow{F} (\mathbb{C}', p')$ , the functor  $F/\mathbf{Fam} : \mathbb{C}/P_p X \rightarrow \mathbb{C}'/P_{p'} X'$  maps  $Z \rightarrow P_p X$  to  $FZ \rightarrow F P_p X \cong P_{p'} X'$ .*

**Corollary 4.8.** *The mapping  $\Gamma \mapsto (\llbracket T\Gamma \rrbracket \rightarrow \mathbf{Fam})$  extends to a **CartExp**-functor from  $\text{FinGat}$  to  $\text{CAT}/\mathbf{Fam}$  as the composition of*

<sup>3</sup>In other words,  $f$  is in the splitting cleavage of the fibration  $\mathbf{Fam} \rightarrow \mathbf{Set}$  mapping  $(U, El)$  to  $U$ .

the two-sortification functor  $T$  and the image of the model functor by  $-/\text{Fam}$ .

Similarly, we may hope to decompose  $\Gamma \mapsto (F_\Gamma/\text{cFam} \rightarrow \text{Fam})$  as the composition of the bi-initial **CartExp**-functor to  $\text{CAT} // \text{Fam}$ , mapping  $\Gamma$  to  $F_\Gamma: \llbracket \Gamma \rrbracket \rightarrow \text{Fam}$ , and a suitable **CartExp**-functor from  $\text{CAT} // \text{Fam}$  to  $\text{CAT}/\text{Fam}$ , mapping  $F: \mathbb{C} \rightarrow \text{Fam}$  to  $F/\text{cFam} \rightarrow \text{Fam}$ . However, there is an issue when defining the action of the latter desired functor on morphisms. Indeed, the image of a morphism  $\alpha: F' \circ G \rightarrow F$  between  $F: \mathbb{C} \rightarrow \text{Fam}$  and  $F': \mathbb{C}' \rightarrow \text{Fam}$  in  $\text{CAT} // \text{Fam}$  must be a functor  $F'/\text{cFam} \rightarrow F/\text{cFam}$  over  $\text{Fam}$ . An obvious attempt is to define it as mapping a cartesian morphism  $F(M) \rightarrow (U, El)$  to the composite  $F'(G(M)) \xrightarrow{\alpha_M} F(M) \rightarrow (U, El)$ . Unfortunately, this composite is not necessarily cartesian, so this mapping does not, in general, land in  $F'/\text{cFam}$ . However, it does if  $\alpha_M$  is cartesian, suggesting the following definition.

**Definition 4.9.** Let  $\text{CAT} //_{\text{c}} \text{Fam}$  denote the wide subcategory of  $\text{CAT} // \text{Fam}$  with the same objects, but restricting to (pointwise) cartesian morphisms, i.e. to morphisms  $\alpha: F' \circ G \rightarrow F$  that for every object  $c$  of  $\mathbb{C}$ , the component  $\alpha_c$  is a cartesian morphism in  $\text{Fam}$ .

The subcategory  $\text{CAT} //_{\text{c}} \text{Fam}$  actually inherits the **CartExp**-structure of  $\text{CAT} // \text{Fam}$ .

**Proposition 4.10.** *The category  $\text{CAT} //_{\text{c}} \text{Fam}$  is a sub-**CartExp**-category of  $\text{CAT} // \text{Fam}$ , in the sense that it has finite limits, the exponentiable morphism from Theorem 4.3 is in  $\text{CAT} //_{\text{c}} \text{Fam}$  and is again exponentiable, and finally, the functor  $\text{CAT} //_{\text{c}} \text{Fam} \rightarrow \text{CAT} // \text{Fam}$  preserves finite limits and pushforwards along this exponentiable morphism.*

Now we can define the desired **CartExp**-functor from  $\text{CAT} //_{\text{c}} \text{Fam}$  to  $\text{CAT}/\text{Fam}$ .

**Proposition 4.11.** *The mapping  $(\mathbb{C} \xrightarrow{F} \text{Fam}) \mapsto (F/\text{cFam} \rightarrow \text{Fam})$  extends to a **CartExp**-functor from  $\text{CAT} //_{\text{c}} \text{Fam}$  to  $\text{CAT}/\text{Fam}$ .*

**Proposition 4.12.** *There is a strict bi-initial **CartExp**-functor from  $\text{FinGat}$  to  $\text{CAT} //_{\text{c}} \text{Fam}$  that maps a theory  $\Gamma$  to a functor  $\text{Fam}_\Gamma: \llbracket \Gamma \rrbracket \rightarrow \text{Fam}$  which is isomorphic to the family functor of  $\Gamma$  defined in Corollary 4.4. Moreover, this isomorphism is natural in  $\Gamma$ .*

**PROOF.** By uniqueness (up to isomorphism) of the initial **CartExp**-functor, the composition  $\text{FinGat} \rightarrow \text{CAT} //_{\text{c}} \text{Fam} \rightarrow \text{CAT} // \text{Fam}$  is isomorphic to the bi-initial **CartExp**-functor from  $\text{FinGat}$  to  $\text{CAT} // \text{Fam}$ .  $\square$

The following corollary concludes the argument for the proof of Theorem 4.5, as bi-initiality implies that the **CartExp**-functor considered here is isomorphic to the one considered in Corollary 4.8.

**Corollary 4.13.** *There is a **CartExp**-functor from  $\text{FinGat}$  to the slice category  $\text{CAT}/\text{Fam}$  that maps a theory  $\Gamma$  to  $\text{Fam}_\Gamma/\text{cFam} \rightarrow \text{Fam}$ , with  $\text{Fam}_\Gamma: \llbracket \Gamma \rrbracket \rightarrow \text{Fam}$  being isomorphic to the family functor of  $\Gamma$ , and this isomorphism is natural in  $\Gamma$ . This **CartExp**-functor is the composition of the bi-initial **CartExp**-functor to  $\text{CAT} //_{\text{c}} \text{Fam}$  defined in Proposition 4.12, mapping  $\Gamma$  to  $\text{Fam}_\Gamma: \llbracket \Gamma \rrbracket \rightarrow \text{Fam}$ , and the **CartExp**-functor from  $\text{CAT} //_{\text{c}} \text{Fam}$  to  $\text{CAT}/\text{Fam}$ , mapping  $F: \mathbb{C} \rightarrow \text{Fam}$  to  $F/\text{cFam} \rightarrow \text{Fam}$ .*

### 4.3 Recovering the Right Adjoint from the Coreflector Morphism

In this section, we show that the right adjoint of the strict coreflection between  $\llbracket T\Gamma \rrbracket$  and  $\llbracket \Gamma \rrbracket$  constructed in Corollary 4.6 is (up to isomorphism) the image of the coreflector morphism of  $\Gamma$  by the model functor.

The proof consists in exploiting Theorem 2.13 to conclude that the two below natural transformations from  $\llbracket T- \rrbracket$  to  $\llbracket - \rrbracket$  are identical, where  $\alpha_\Gamma$  is the image of the coreflector morphism  $T\Gamma \rightarrow \Gamma$  by the model functor, and  $\beta_\Gamma$  is, up to isomorphism, the right adjoint of the coreflection constructed in Corollary 4.6.

$$\begin{array}{ccc} \llbracket - \rrbracket \circ T & \xrightarrow{\quad} & \text{CAT}/\text{Fam} \\ & \searrow \alpha \quad \swarrow \beta & \\ \text{FinGat} & \xrightarrow{\quad [-] \quad} & \text{CAT} \end{array}$$

**Proposition 4.14.** *The families of morphisms  $(\alpha_\Gamma)_\Gamma$  and  $(\beta_\Gamma)_\Gamma$  are natural. Moreover, their naturality squares for  $p_{\mathcal{G}}$  are both the right pullback of Equation (1), for  $\mathbb{C} = \text{CAT}$ .*

We finally get the equality of  $\alpha$  and  $\beta$  by a direct application of Theorem 2.13.

**Corollary 4.15.** *The natural transformations  $\alpha$  and  $\beta$  are equal. In particular, the image of the coreflector morphism by the model functor coincide (up to isomorphism) with the right adjoint of the coreflection of Corollary 4.6.*

## 5 Initial Models of GATs

In this section, we provide a direct proof that any theory has an initial model, exploiting the bi-initiality property of  $\text{FinGat}$ .

The first step consists in constructing a suitable model of a theory  $\Omega$ . As in [14], we exploit the Yoneda lemma [22, §III.2]: an object of  $\llbracket \Omega \rrbracket$  is equivalently given by a natural transformation from the representable functor  $\text{hom}(\Omega, -)$  to the model functor. The point is that we can use Theorem 2.13 to construct such a natural transformation. Before, we introduce some convenient type-theoretic notations to denote morphisms to  $X_{\mathcal{G}}$  and  $Y_{\mathcal{G}}$  in  $\text{FinGat}$ .

In the rest of this section, we assume given a theory  $\Omega$ , the goal being to construct an initial model of it.

**Notation 5.1.** Given  $A: \Omega \rightarrow X_{\mathcal{G}}$ , we denote the set of morphisms  $t: \Omega \rightarrow Y_{\mathcal{G}}$  such that  $p_{\mathcal{G}} \circ t = A$  by  $\text{Tm}(A)$ . The notation is motivated by the fact that a morphism  $\Omega \rightarrow X_{\mathcal{G}}$  corresponds to a term  $\Omega \vdash A: \text{Set}$ , and a morphism  $\Omega \rightarrow Y_{\mathcal{G}}$  above  $X_{\mathcal{G}}$  corresponds to a term  $\Omega \vdash t: A$ .

Given a model  $\omega$  of  $\Omega$  and element  $t \in \text{Tm}(A)$ , the pointed set  $\llbracket t \rrbracket(\omega)$  is necessarily  $\llbracket A \rrbracket(\omega)$ , with a distinguished element denoted by  $[t](\omega)$ .

As a direct consequence of the functoriality of  $\llbracket t \rrbracket$ , we get naturality of  $[t](\omega)$  in  $\omega$ :

**Lemma 5.2.** *Given any morphism of models  $f: \omega \rightarrow \omega'$ , we have  $\llbracket A \rrbracket(f)([t](\omega)) = [t](\omega')$ .*

We now apply Theorem 2.13 to construct our candidate initial model, using the above notations.

**Proposition 5.3.** *There is a unique model  $0_\Omega$  of  $\Omega$  such that,*



- for any  $A: \Omega \rightarrow X_{\mathcal{G}}$ , the set  $\llbracket A \rrbracket(0_{\Omega})$  is  $\text{Tm}(A)$ ;
- for any  $t \in \text{Tm}(A)$ , the element  $\llbracket t \rrbracket(0_{\Omega})$  of  $\llbracket A \rrbracket(0_{\Omega})$  is  $t$  itself.

PROOF. Any model  $\omega$  induces a natural transformation from  $\text{hom}(\Omega, -)$  to the model functor such that  $\text{hom}(\Omega, \Gamma) \rightarrow \llbracket \Gamma \rrbracket$  maps a morphism  $\sigma: \Omega \rightarrow \Gamma$  to  $\llbracket \sigma \rrbracket(\omega)$ . In this respect, the two conditions determines the following naturality square for  $p_{\mathcal{G}}$ .

$$\begin{array}{ccc} \text{hom}(\Omega, Y_{\mathcal{G}}) & \xrightarrow{\quad} & \mathbf{PtdSet} \\ \downarrow & \lrcorner & \downarrow \\ \text{hom}(\Omega, X_{\mathcal{G}}) & \xrightarrow{\quad} & \mathbf{Set} \end{array} \quad (2)$$

By Theorem 2.13, there is a unique natural transformation from  $\text{hom}(\Omega, -)$  to the model functor, and by the Yoneda lemma, it corresponds to a unique model  $0_{\Omega}$  of  $\Omega$ , recovered from the natural transformation by evaluating it at  $\text{id}_{\Omega} \in \text{hom}(\Omega, \Omega)$ .  $\square$

Next, we assume given a model  $\omega$  of  $\Omega$ . To prove that  $0_{\Omega}$  is initial, we need to construct a morphism from  $0_{\Omega}$  to  $\omega$  and show that it is unique. Note that a morphism in  $\llbracket \Omega \rrbracket$  is an object of its category  $\llbracket \Omega \rrbracket^{\rightarrow}$  of arrows. This suggests using a similar strategy to construct a natural transformation from  $\text{hom}(\Omega, -)$  to  $\llbracket - \rrbracket^{\rightarrow}$ . To apply Theorem 2.13, we need  $\mathbf{FinGat} \xrightarrow{\llbracket - \rrbracket} \mathbf{CAT} \xrightarrow{(-)^{\rightarrow}} \mathbf{CAT}$  to be a **CartExp**-functor. By definition, the model functor is a **CartExp**-functor, so we only need to ensure that  $(-)^{\rightarrow}$  is. But clearly,  $(-)^{\rightarrow}$  does not map the exponentiable functor  $p_{\mathbf{CAT}}: \mathbf{PtdSet} \rightarrow \mathbf{Set}$  to itself, so it cannot be a **CartExp**-functor, unless we consider the codomain  $\mathbf{CAT}$  with a different **CartExp**-structure, choosing the exponentiable morphism to be precisely the image of  $p_{\mathbf{CAT}}$  by  $(-)^{\rightarrow}$ . Unfortunately,  $(-)^{\rightarrow}: \mathbf{CAT} \rightarrow \mathbf{CAT}$  is not a **CartExp**-functor between  $(\mathbf{CAT}, p_{\mathbf{CAT}})$  and  $(\mathbf{CAT}, p_{\mathbf{CAT}}^{\rightarrow})$ , because it does not preserve pushforwards along  $\mathbf{PtdSet} \rightarrow \mathbf{Set}$ . However, the following variant works.

**Proposition 5.4.** *Let  $\text{dom}_{-}: \mathbf{CAT} \rightarrow \mathbf{CAT}^{\rightarrow}$  denote the functor mapping a category  $C$  to the codomain functor  $\text{cod}_{C}: C^{\rightarrow} \rightarrow C$ . This induces a **CartExp**-functor between the **CartExp**-categories  $(\mathbf{CAT}, p_{\mathbf{CAT}})$  and  $(\mathbf{CAT}^{\rightarrow}, \text{cod}_{-}(p_{\mathbf{CAT}}))$ .*

PROOF. The functor  $P_{\text{cod}_{-}(p_{\mathbf{CAT}})}: \mathbf{CAT}^{\rightarrow} \rightarrow \mathbf{CAT}^{\rightarrow}/\text{cod}_{-}(p_{\mathbf{CAT}})$  maps a functor  $F: C \rightarrow D$  to the blue functor below left, equipped with the below right morphism to  $\mathbf{Set}^{\rightarrow} \xrightarrow{\text{cod}} \mathbf{Set}$ , where

- $\mathbf{Set}/\mathbf{Fam}(C)$  denotes the comma category of  $\mathbf{Set} \xrightarrow{\text{id}} \mathbf{Set}$  and  $\mathbf{Fam}(C) \rightarrow \mathbf{Set}$ : objects are triples  $(X, (c_y)_{y \in Y}, X \xrightarrow{f} Y)$  of a set  $X$ , a family  $(c_y)_{y \in Y}$  of objects of  $C$  and a function  $f: X \rightarrow Y$ ;
- the functor  $\mathbf{Set}/\mathbf{Fam}(C) \rightarrow \mathbf{Fam}(C)$  on the left maps such an object to the family  $(c_{f(x)})_{x \in Y}$ .

$$\begin{array}{ccccc} PF & \xrightarrow{\quad} & \mathbf{Fam}(C) & PF & \xrightarrow{\quad} & \mathbf{Set}/\mathbf{Fam}(C) & \xrightarrow{\quad} & \mathbf{Set}^{\rightarrow} \\ \downarrow & \lrcorner & \downarrow \mathbf{Fam}(F) & \downarrow & \searrow \pi_2 & \downarrow \text{cod} & & \\ \mathbf{Set}/\mathbf{Fam}(C) & \xrightarrow{\quad} & \mathbf{Fam}(C) & & \mathbf{Fam}(C) & \xrightarrow{\quad} & \mathbf{Set} \\ \downarrow \pi_2 & & & & & & \\ \mathbf{Fam}(C) & & & & & & \end{array}$$

It can be checked that this is indeed the pushforward of  $F$  along  $\text{cod}_{-}(p_{\mathbf{CAT}})$ , and moreover, that  $\text{cod}_{-}$  preserves pushforwards along  $p_{\mathbf{CAT}}$ .  $\square$

Now we are able to construct a unique morphism from  $0_{\Omega}$  to  $\omega$  of  $\Omega$ , exploiting Theorem 2.13 to specify a suitable natural transformation between the functor  $\mathbf{FinGat} \rightarrow \mathbf{CAT}^{\rightarrow}$  mapping  $\Gamma$  to the function  $\llbracket - \rrbracket(\omega): \text{hom}(\Omega, \Gamma) \rightarrow \llbracket \Gamma \rrbracket$ , and  $\mathbf{FinGat} \xrightarrow{\llbracket - \rrbracket} \mathbf{CAT} \xrightarrow{\text{cod}_{-}} \mathbf{CAT}^{\rightarrow}$ .

**Proposition 5.5.** *There exists a unique morphism  $i$  from  $0_{\Omega}$  to  $\omega$  in  $\llbracket \Omega \rrbracket$  such that, for any  $A: \Omega \rightarrow X_{\mathcal{G}}$ , the function  $\llbracket A \rrbracket(i): \llbracket A \rrbracket(0_{\Omega}) \rightarrow \llbracket A \rrbracket(\omega)$  maps  $t \in \text{Tm}(A)$  to  $\llbracket t \rrbracket(\omega)$ .*

PROOF. The proof is similar to that of Proposition 5.3, but we additionally need to ensure that the morphism induced by the natural transformation  $\text{hom}(\Omega, -) \rightarrow \llbracket - \rrbracket^{\rightarrow}$  by the Yoneda lemma is indeed a morphism between  $0_{\Omega}$  and  $\omega$ . For the domain, this follows from Proposition 5.3. For the codomain, this is by definition of a morphism in  $\mathbf{CAT}^{\rightarrow}$ .  $\square$

Uniqueness of the morphism comes from the following simple observation.

**Lemma 5.6.** *Any morphism from  $0_{\Omega}$  to  $\omega$  satisfies the condition of Proposition 5.5.*

PROOF. Consider a morphism  $f: 0_{\Omega} \rightarrow \omega$ . By Lemma 5.2, we have  $\llbracket A \rrbracket(f)(\llbracket t \rrbracket(0_{\Omega})) = \llbracket t \rrbracket(\omega)$ . By Proposition 5.3, we have  $\llbracket t \rrbracket(0_{\Omega}) = t$ , so the conclusion follows.  $\square$

**Corollary 5.7.** The model  $0_{\Omega}$  is initial in  $\llbracket \Omega \rrbracket$ : there exists a unique morphism from  $0_{\Omega}$  to any model  $\omega$  of  $\Omega$ .

## 6 Two-sortification is Fully Faithful

In this section, we show that the two-sortification functor  $T$  from  $\mathbf{FinGat}$  to  $\mathbf{FinGat}/\mathbf{Fam}$  is fully faithful. We mostly rely on the following proposition.

**Proposition 6.1.** *Let  $\mathbb{C}$  be a **CartExp**-category and  $I: \mathbf{FinGat} \rightarrow \mathbb{C}$  be a **CartExp**-functor. Then,  $I$  is faithful (resp. fully faithful) if and only if  $I$  is faithful (resp. fully faithful) at  $X_{\mathcal{G}}$  and  $Y_{\mathcal{G}}$ , i.e., for any theory  $\Omega$ , the functions  $\text{hom}(\Omega, X_{\mathcal{G}}) \rightarrow \text{hom}(I\Omega, IX_{\mathcal{G}})$  and  $\text{hom}(\Omega, Y_{\mathcal{G}}) \rightarrow \text{hom}(I\Omega, IY_{\mathcal{G}})$  are injective (resp. bijective).*

PROOF. We focus on the hard direction, which is the "if" part. We show that  $\mathbf{FinGat}$  is equal to its full subcategory  $S$  spanned by theories  $\Gamma$  such that  $I$  is faithful (resp. fully faithful) at  $\Gamma$ . Note that  $S$  includes  $X_{\mathcal{G}}$  and  $Y_{\mathcal{G}}$  by assumption. It is easy to check that  $S$  is stable under limits preserved by  $I$ , and that if  $\Gamma$  is in  $S$ , then so is  $P\Gamma$ . Therefore,  $S$  is a full **CartExp**-subcategory of  $\mathbf{FinGat}$ . By bi-initiality of  $\mathbf{FinGat}$ , we get a functor from  $\mathbf{FinGat}$  to  $S$  such that the inclusion composed with this functor is isomorphic to the identity. This entails that any GAT is isomorphic to a theory at which  $I$  is faithful (resp. fully faithful). The claimed result easily follows.  $\square$

Proposition 6.1 requires the morphism in  $\mathbf{Set}^{\rightarrow}$  from  $\text{hom}(\Omega, X_{\mathcal{G}}) \rightarrow \text{hom}(\Omega, Y_{\mathcal{G}})$  to  $\text{hom}(I\Omega, IX_{\mathcal{G}}) \rightarrow \text{hom}(I\Omega, IY_{\mathcal{G}})$  induced by the functorial action of  $I$  to be a monomorphism or an isomorphism. It will prove useful to rephrase this condition through the equivalence  $\mathbf{Set}^{\rightarrow} \simeq \mathbf{Fam}$ .

**Notation 6.2.** Let  $I: \mathbf{FinGat} \rightarrow \mathbb{C}$  be a **CartExp**-functor and  $\Omega$  be a theory. The image of  $\text{hom}(I\Omega, IX_{\mathcal{G}}) \rightarrow \text{hom}(I\Omega, IY_{\mathcal{G}})$  by the equivalence  $\mathbf{Set}^{\rightarrow} \simeq \mathbf{Fam}$  is denoted by  $\mathcal{F}_I(\Omega)$ , omitting  $I$  when it is the identity functor on  $\mathbf{FinGat}$ . Explicitly,  $\mathcal{F}_I(\Omega)$  is the family  $(\text{Tm}_I(A))_{A \in \text{hom}(I\Omega, IX_{\mathcal{G}})}$ , where  $\text{Tm}_I(A)$  denotes the set of morphisms  $t: I\Omega \rightarrow IY_{\mathcal{G}}$  such that  $Ip_{\mathcal{G}} \circ t = A$ .

**Corollary 6.3.** Let  $I: \mathbf{FinGat} \rightarrow \mathbb{C}$  be a **CartExp**-functor. Then,  $I$  is faithful (resp. fully faithful) if and only if, for any theory  $\Omega$ , the family morphism  $\mathcal{F}(\Omega) \rightarrow \mathcal{F}_I(\Omega)$  induced by the functorial action of  $I$  is a monomorphism (resp. isomorphism).

Using these results, in Section 6.1, we show that  $T$  is faithful, and in Section 6.2, we show that it is also full.

## 6.1 Faithfulness

In this section, we show that  $\llbracket T- \rrbracket$  is faithful, and therefore, so is  $T$ . We will rather work with the following isomorphic variant of  $\llbracket T- \rrbracket$  for which we have an exact description of the images of  $X_{\mathcal{G}}$  and  $Y_{\mathcal{G}}$ .

**Notation 6.4.** We denote the **CartExp**-functor  $\mathbf{FinGat} \rightarrow \mathbf{CAT}/\mathbf{Fam}$  from Corollary 4.13 mapping a theory  $\Gamma$  to  $\text{Fam}_{\Gamma}/_c \mathbf{Fam}$  by  $\mathbb{T}$ .

Recall that  $\mathbb{T}$  is isomorphic  $\llbracket T- \rrbracket$  by Theorem 2.7. Now, for any theory  $\Omega$ , we construct an object of  $\mathbb{T}\Omega$  such that given a pair of morphisms  $f, g$  from  $\Omega$  to  $X_{\mathcal{G}}$  (resp.  $Y_{\mathcal{G}}$ ), if its image by  $\mathbb{T}f$  and  $\mathbb{T}g$  are equal, then  $f$  and  $g$  are equal as well. This ensures that  $\mathbb{T}$ , and therefore  $\llbracket T- \rrbracket$ , is faithful at  $X_{\mathcal{G}}$  and  $Y_{\mathcal{G}}$ .

**Proposition 6.5.** For any theory  $\Omega$ , there is an object  $M_{\Omega}$  of  $\mathbb{T}\Omega$  such that

- $!(M_{\Omega})$  is the family  $\mathcal{F}(\Omega)$ , where  $!: \mathbb{T}\Omega \rightarrow \mathbf{Fam}$  is the canonical projection mapping any object  $\text{Fam}_{\Omega}(\omega) \rightarrow A$  to  $A$ ;
- for any  $A: \Omega \rightarrow X_{\mathcal{G}}$ , the cartesian family morphism  $\mathbb{T}A(M_{\Omega})$  is the morphism from  $\text{Fam}_{X_{\mathcal{G}}}(\text{Tm}(A)) = (\text{Tm}(A))_{* \in \{*\}}$  to  $\mathcal{F}(\Omega)$  mapping  $*$  to  $A \in \text{hom}(\Omega, X_{\mathcal{G}})$ ;
- for any  $(\Omega \xrightarrow{t} Y_{\mathcal{G}}) \in \text{Tm}(A)$ , the cartesian family morphism  $\mathbb{T}t(M_{\Omega})$  is the same morphism  $\text{Fam}_{Y_{\mathcal{G}}}(1 \xrightarrow{t} \text{Tm}(A)) = (\text{Tm}(A))_{* \in \{*\}}$  to  $\mathcal{F}(\Omega)$  as above, where  $A$  denotes  $p_{\mathcal{G}} \circ t$ .

**PROOF.** The proof is similar to Proposition 5.3: we apply Theorem 2.13 to construct a natural transformation between

- the functor from  $\mathbf{FinGat}$  to  $\mathbf{CAT}/\mathbf{Fam}$  that maps any theory  $\Gamma$  to the constant functor  $\text{hom}(\Omega, \Gamma) \rightarrow \mathbf{Fam}$  mapping any morphism to  $\mathcal{F}(\Omega)$ , and
- the functor  $\mathbb{T}$ .

The Yoneda lemma yields the desired object  $M_{\Omega}$  of  $\mathbb{T}\Omega$ .  $\square$

**Corollary 6.6.**  $\llbracket T- \rrbracket$  is faithful, and thus, so is  $T$ .

**Remark 6.7.** Corollary 6.6 raises the question whether the model functor is faithful as well. We leave this question open.

## 6.2 Fullness

In this section, we show that  $T$  is also full. Since we showed in the previous section that  $T$  is faithful, by Corollary 6.3, we know that  $\mathcal{F}(\Omega) \rightarrow \mathcal{F}_T(\Omega)$  is a monomorphism for any theory  $\Omega$ . Thus, it is enough to construct a section. We start by replacing  $\mathcal{F}_T(\Omega)$  with an isomorphic family.

**Notation 6.8.** Given a theory  $\Omega$ , we denote the composition  $T\Omega \rightarrow PX_{\mathcal{G}} \rightarrow X_{\mathcal{G}}$  by  $U_{\Omega}$ . Given  $(T\Omega \xrightarrow{A} Y_{\mathcal{G}}) \in \text{Tm}(U_{\Omega})$ , we denote the composition  $\Omega \rightarrow TX_{\mathcal{G}} \xrightarrow{\varepsilon_{X_{\mathcal{G}}}} X_{\mathcal{G}}$  by  $\text{El } A$ , where the first morphism is the one corresponding to  $A$  and  $U_{\Omega}$  by the universal property of the pullback (1) defining  $TX_{\mathcal{G}}$ , and  $\varepsilon_{X_{\mathcal{G}}}$  is the counit at  $X_{\mathcal{G}}$  of the adjunction  $p_{\mathcal{G}}^* \dashv P$ .

**Remark 6.9.** The notations are motivated by the fact that an element of  $\text{Tm}(U_{\Omega})$  corresponds to a term  $T\Omega \vdash A: U$ , and an element of  $\text{Tm}(\text{El } A)$  corresponds to a term  $T\Omega \vdash t: \text{El } A$ .

**Proposition 6.10.** Given a theory  $\Omega$ , let  $\mathcal{E}(\Omega)$  denote the family  $(\text{Tm}(\text{El } A))_{A \in \text{Tm}(U_{\Omega})}$ . Then, the morphism from  $\mathcal{F}_T(\Omega)$  to  $\mathcal{E}(\Omega)$  mapping  $A \in \text{hom}(T\Omega, TX_{\mathcal{G}})$  to  $T\Omega \xrightarrow{TA} TX_{\mathcal{G}} \rightarrow Y_{\mathcal{G}}$  and  $(T\Omega \xrightarrow{t} TY_{\mathcal{G}}) \in \text{Tm}(A)$  to  $T\Omega \xrightarrow{t} TY_{\mathcal{G}} \rightarrow Y_{\mathcal{G}}$  is an isomorphism, where  $TX_{\mathcal{G}} \rightarrow X_{\mathcal{G}}$  and  $TY_{\mathcal{G}} \rightarrow Y_{\mathcal{G}}$  are the projections from the pullbacks (1) defining  $TX_{\mathcal{G}}$  and  $TY_{\mathcal{G}}$ .

**PROOF.** This is a consequence of the definitions of  $TX_{\mathcal{G}}$  and  $TY_{\mathcal{G}}$  as suitable pullbacks.  $\square$

Thanks to this isomorphism, it is enough to construct a section of  $\mathcal{F}(\Omega) \rightarrow \mathcal{F}_T(\Omega) \xrightarrow{\cong} \mathcal{E}(\Omega)$ . We exploit initiality of the following object of  $\mathbb{T}\Omega$ .

**Proposition 6.11.** For any theory  $\Omega$ , there is an initial object  $Z_{\Omega}$  of  $\mathbb{T}\Omega$  such that:

- $!(Z_{\Omega})$  is the family  $\mathcal{E}(\Omega)$ ;
- for any  $A: \Omega \rightarrow X_{\mathcal{G}}$ , the cartesian family morphism  $\mathbb{T}A(Z_{\Omega})$  is the morphism from  $(\text{Tm}(\text{El } A'))_{* \in \{*\}}$  to  $\mathcal{E}(\Omega)$  mapping  $*$  to  $A'$ , where  $A'$  denotes  $(T\Omega \xrightarrow{TA} TX_{\mathcal{G}} \rightarrow Y_{\mathcal{G}})$ ;
- for any  $(\Omega \xrightarrow{t} Y_{\mathcal{G}}) \in \text{Tm}(A)$ , the cartesian family morphism  $\mathbb{T}t(M_{\Omega})$  is the same as above, but where  $\text{Tm}(A')$  is considered pointed by  $T\Omega \xrightarrow{Tt} TY_{\mathcal{G}} \rightarrow Y_{\mathcal{G}}$ .

**PROOF.** We take  $Z_{\Omega}$  to be the initial model  $0_{T\Omega}$  of  $T\Omega$  from Proposition 5.3, through the isomorphism  $\llbracket T\Omega \rrbracket \simeq \mathbb{T}\Omega$  from Theorem 2.7. The result follows by reasoning about the natural transformation  $(\text{hom}(T\Omega, \Gamma) \rightarrow \llbracket \Gamma \rrbracket)_{\Gamma}$  mapping  $\sigma$  to  $\llbracket \sigma \rrbracket 0_{T\Omega}$ . More details are available in the appendix.  $\square$

We conclude by the following result.

**Proposition 6.12.** The image of the initial morphism  $Z_{\Omega} \rightarrow M_{\Omega}$  by  $!$  yields a family morphism which is a section of the morphism  $\mathcal{F}(\Omega) \rightarrow \mathcal{F}_T(\Omega) \cong \mathcal{E}(\Omega)$ .

**PROOF.** We deduce the proposition from the uniqueness result of Theorem 2.13 to show equality of two suitable natural transformations between functors from  $\mathbf{FinGat}$  to the **CartExp**-category  $\mathbf{CAT} // \mathbf{Fam}$  from Theorem 4.3. The domain of these natural transformations is the functor  $\mathbf{FinGat} \rightarrow \mathbf{CAT} // \mathbf{Fam}$  mapping  $\Gamma$  to the constant function  $\text{hom}(\Omega, \Gamma) \rightarrow \text{Fam}(\mathbf{Set})$  that maps any morphism to  $\mathcal{E}(\Omega)$ . The codomain is the strict initial functor  $\mathbf{FinGat} \rightarrow \mathbf{CAT} // _c \mathbf{Fam}$  from Proposition 4.12 composed with the embedding  $\mathbf{CAT} // _c \mathbf{Fam} \rightarrow \mathbf{CAT} // \mathbf{Fam}$  from Proposition 4.10, mapping  $\Gamma$  to  $\text{Fam}_{\Gamma}: \llbracket \Gamma \rrbracket \rightarrow \text{Fam}(\mathbf{Set})$ .

The first natural transformation consists of the function from  $\text{hom}(\Omega, \Gamma)$  to  $\llbracket \Gamma \rrbracket$  mapping any morphism  $\sigma$  to  $\llbracket \sigma \rrbracket(R_\Omega Z_\Omega)$ , and the family morphism  $\text{Fam}_\Gamma \llbracket \sigma \rrbracket R_\Omega Z_\Omega \xrightarrow{\mathbb{T}\sigma(Z_\Omega)} \mathcal{E}(\Omega)$ . The second natural transformation is like the first one, but we postcompose the family morphism with  $\mathcal{E}(\Omega) \rightarrow \mathcal{F}(\Omega) \rightarrow \mathcal{E}(\Omega)$ . We now show that the two natural transformations are equal, thus recovering the desired equality by taking  $\sigma = \text{Id}_\Omega$ , since  $\mathbb{T}\text{Id}_\Omega(Z_\Omega) = \mathbb{T}Z_\Omega$  is necessarily an isomorphism as  $Z_\Omega$  is initial. To that end, we apply Theorem 2.13. The fact that the naturality squares at  $p_{\mathcal{G}}$  are pullbacks reduces to the fact that the naturality square at  $p_{\mathcal{G}}$  after whiskering by  $\text{CAT} // \text{Fam} \rightarrow \text{CAT}$  are pullbacks. This is deduced from the fact that  $RZ_\Omega$  is initial and thus is isomorphic to the initial model of  $\Omega$  of Proposition 5.3, which satisfies this pullback property by construction.

It remains to show that the above family morphisms are equal for any morphism  $\sigma$  from  $\Omega$  to  $X_{\mathcal{G}}$  or  $Y_{\mathcal{G}}$ . We focus on the case of  $X_{\mathcal{G}}$ , the case of  $Y_{\mathcal{G}}$  being similar. Let  $\sigma: \Omega \rightarrow X_{\mathcal{G}}$  be a morphism of GATs. By applying  $\mathbb{T}\sigma$  to the initial morphism  $i: Z_\Omega \rightarrow M_\Omega$ , we get that  $\text{Fam}_\Gamma \llbracket \sigma \rrbracket R_\Omega Z_\Omega \xrightarrow{\mathbb{T}\sigma(Z_\Omega)} \mathcal{E}(\Omega) \xrightarrow{!i} \mathcal{F}(\Omega)$  is equal to the following composition.

$$\text{Fam}_\Gamma \llbracket \sigma \rrbracket R_\Omega Z_\Omega \xrightarrow{\text{Fam}_\Gamma \llbracket \sigma \rrbracket R_\Omega i} \text{Fam}_\Gamma \llbracket \sigma \rrbracket R_\Omega M_\Omega \xrightarrow{\mathbb{T}\sigma(M_\Omega)} \mathcal{F}(\Omega)$$

It is easy to check that the two family morphisms are equal on the base sets, using the characterisations of  $Z_\Omega$  and  $M_\Omega$ . Now, for the unique fiber  $\text{Tm}(\sigma)$  of the domain, consider an element  $(\Omega \xrightarrow{t} Y_{\mathcal{G}}) \in \text{Tm}(\sigma)$ . Then, by considering  $\llbracket t \rrbracket R_\Omega i$ , we get that the image of  $t$  by  $\text{Fam}_\Gamma \llbracket \sigma \rrbracket R_\Omega i$  is  $(T\Omega \xrightarrow{Tt} TY_{\mathcal{G}} \rightarrow Y_{\mathcal{G}})$ . It follows that the two family morphisms are equal on that fiber as well.  $\square$

**Corollary 6.13.** *T is fully faithful.*

**PROOF.** By Corollary 6.3, it is enough to show that for any theory  $\Omega$ , the family morphism  $\mathcal{F}(\Omega) \rightarrow \mathcal{F}_T(\Omega)$  is an isomorphism. We already showed that it is a monomorphism in Corollary 6.6, and Proposition 6.12 provides a section. Therefore, it is an isomorphism and we get the desired result.  $\square$

## 7 Infinite GATs

In this section, we sketch how to adapt our results to GATs that are not necessarily finite. One could think of them as infinite contexts, following the type-theoretic description of finite GATs, or more formally, as Pro-objects [6, Section 2.2] of  $\text{FinGat}$ , that is, as small cofiltered diagrams of finite GATs. For example, the infinite context  $A_1 : \text{Set}, A_2 : A_1 \rightarrow \text{Set}, \dots$  is modeled as a functor from the ordinal  $\omega^{\text{op}}$ , viewed as a category, to  $\text{FinGat}$ , mapping  $n$  to the finite theory  $A_1 : \text{Set}, A_2 : A_1 \rightarrow \text{Set}, \dots, A_n : \dots \rightarrow \text{Set}$ .

The main result of this section is the following statement.

**Theorem 7.1.** *The category  $\text{Gat}$  of GATs, equipped with  $p_{\mathcal{G}}$  viewed as a morphism in  $\text{Gat}$ , is bi-initial in the 2-category which is like  $\text{CartExp}$  except that objects are now categories with all limits and the morphisms must preserve them as functors. Explicitly:*

- an object is a category  $\mathbb{C}$  with all limits equipped with an exponentiable morphism  $p: Y \rightarrow X$  in  $\mathbb{C}$ ;

- a morphism from  $(\mathbb{C}, p)$  to  $(\mathbb{C}', p')$  is a limit-preserving functor  $F: \mathbb{C} \rightarrow \mathbb{C}'$  equipped with an isomorphism  $F(p) \cong p'$ , preserving pushforwards along  $p$ ;
- a 2-cell between two morphisms is a natural transformation between the underlying functors.

The proofs and definitions of the previous sections can be easily adapted to this bi-initiality property. In particular, we still get a model functor and initial models, a fully faithful two-sortification functor from  $\text{Gat}$  to the slice category  $\text{Gat}/\text{Fam}$ , a coreflector morphism whose image by the model functor is the right adjoint of a strict coreflection between models of a GAT and its translation, together with the explicit description of the models of the reduced GAT.

The rest of this section is devoted to the proof of Theorem 7.1, mostly combining standard results about locally finitely presentable categories [1]. The starting point is the following characterisation of the category of GATs.

**Proposition 7.2.** *The embedding  $e: \text{FinGat} \rightarrow \text{Gat}$  of finite GATs into GATs makes  $\text{Gat}$  the free filtered completion (or Pro-completion) of  $\text{FinGat}$ , in the sense that  $\text{Gat}$  has filtered limits and for any category  $\mathbb{D}$  with filtered limits, the precomposition functor  $- \circ e: [\text{Gat}, \mathbb{D}]_{f1} \rightarrow [\text{FinGat}, \mathbb{D}]$  is an equivalence of categories, where  $[\text{Gat}, \mathbb{D}]_{f1}$  denotes the full subcategory of  $[\text{Gat}, \mathbb{D}]$  consisting of functors preserving filtered limits.*

This follows from making formal the above intuitive description of infinite GATs. In the same vein, Uemura [27] states that the category of GATs, with our convention of the direction of morphisms<sup>4</sup>, is equivalent to the opposite category of functors preserving finite limits from  $\text{FinGat}$  to  $\text{Set}$ . The connection between these two characterisations of  $\text{Gat}$  is provided by the following standard construction of free filtered completions.

**Proposition 7.3.** *Let  $\mathbb{C}$  be a small cartesian category. Denoting the full subcategory of functors  $\mathbb{C} \rightarrow \text{Set}$  preserving finite limits by  $\text{Lex}(\mathbb{C})$ , the Yoneda embedding  $y: \mathbb{C} \rightarrow \text{Lex}(\mathbb{C})^{\text{op}}$  mapping  $c$  to  $\text{hom}(\_, c)$  is a free filtered completion.*

**PROOF.** By [23, Proposition 1.2.4] and [17, Proposition 5.39].  $\square$

Proposition 7.2 allows us to conclude that  $\text{Gat}$  is complete.

**Proposition 7.4.** *Given a small cartesian category  $\mathbb{C}$ , any free filtered completion  $i: \mathbb{C} \rightarrow \overline{\mathbb{C}}$  preserves finite limits, and moreover,  $\overline{\mathbb{C}}$  is complete. Therefore,  $e: \text{FinGat} \rightarrow \text{Gat}$  preserves finite limits and  $\text{Gat}$  is complete.*

**PROOF.** It is easy to check that if this is true for one free filtered completion of  $\mathbb{C}$ , then it is true for all such completions of  $\mathbb{C}$ . This is true, in particular, for  $y: \mathbb{C} \rightarrow \text{Lex}(\mathbb{C})^{\text{op}}$  from Proposition 7.3, by [1, Proposition 1.45.(ii)].  $\square$

In the view to proving Theorem 7.1, we need to lift the  $\text{CartExp}$ -structure of  $\text{FinGat}$  to  $\text{Gat}$ . In particular, we need to show that  $e(p_{\mathcal{G}})$  is exponentiable in  $\text{Gat}$ . We notably rely on the following free filtered completion of a slice category.

<sup>4</sup>Recall that it is opposite to Cartmell's original definition.

**Proposition 7.5.** *Let  $\mathbb{C}$  be a small cartesian category and  $j: \mathbb{C} \rightarrow \overline{\mathbb{C}}$  be a free filtered completion. Then, given any object  $X$  of  $\mathbb{C}$ , the canonical functor  $\mathbb{C}/X \rightarrow \overline{\mathbb{C}}/jX$  mapping  $c \rightarrow X$  to its image by  $j$  is a free filtered completion.*

**PROOF.** Again, it is enough to show it for one free filtered completion of  $\mathbb{C}$ . Thus, we consider  $y: \mathbb{C} \rightarrow \text{Lex}(\mathbb{C})^{\text{op}}$  from Proposition 7.3. By [24, Proof of Proposition 2.6], the slice category  $\text{Lex}(\mathbb{C})^{\text{op}}/y_{\mathbb{C}}$  is equivalent to  $\text{Lex}(\mathbb{C}/c)^{\text{op}}$ . Explicitly, the equivalence from  $\text{Lex}(\mathbb{C}/c)$  to  $y_{\mathbb{C}}/\text{Lex}(\mathbb{C})$  maps  $F: (\mathbb{C}/c) \rightarrow \text{Set}$  to the functor  $F': \mathbb{C} \rightarrow \text{Set}$  mapping  $d$  to  $F(d \times c \xrightarrow{\pi_2} c)$ , equipped with the morphism  $y_{\mathbb{C}} \rightarrow F'$  given, by the Yoneda lemma, by the element corresponding to the image of the single element of the (terminal) set  $F(c \rightarrow c)$  by  $F\Delta: F(c \rightarrow c) \rightarrow F'(c \times c \xrightarrow{\pi_2} c)$ . From this description, it is clear that  $\mathbb{C} \xrightarrow{y} \text{Lex}(\mathbb{C})^{\text{op}} \simeq \text{Lex}(\mathbb{C})^{\text{op}}/y_{\mathbb{C}}$  is, up to isomorphism, the claimed functor.  $\square$

To show exponentiability of  $e(p_{\mathcal{G}})$  in  $\text{Gat}$ , we need to define a right adjoint to the pullback functor. The following proposition gives a general construction of right adjoints, exploiting the universal property of free filtered completions.

**Proposition 7.6.** *Let  $i: \mathbb{C} \rightarrow \overline{\mathbb{C}}$  and  $j: \mathbb{D} \rightarrow \overline{\mathbb{D}}$  be free filtered completions of small cartesian categories, with a left adjoint  $L: \mathbb{C} \rightarrow \mathbb{D}$  and a functor  $\bar{L}: \overline{\mathbb{C}} \rightarrow \overline{\mathbb{D}}$  preserving filtered limits, related by an isomorphism  $j \circ L \cong \bar{L} \circ i$ . Then,  $\bar{L}$  has a right adjoint  $\bar{R}: \overline{\mathbb{D}} \rightarrow \overline{\mathbb{C}}$  such that the mate  $i \circ R \rightarrow \bar{R} \circ j$  of the above isomorphism is an isomorphism as well, where  $R: \mathbb{D} \rightarrow \mathbb{C}$  denotes the right adjoint of  $L$ .*

**Corollary 7.7.** *Given a free filtered completion  $i: \mathbb{C} \rightarrow \overline{\mathbb{C}}$  of a cartesian category  $\mathbb{C}$  and an exponentiable morphism  $p: Y \rightarrow X$  in  $\mathbb{C}$ , the morphism  $ip: iY \rightarrow iX$  is exponentiable in  $\overline{\mathbb{C}}$  and  $i$  preserves pushforwards along  $p$ .*

**PROOF.** We apply Proposition 7.6 by taking  $i$  and  $j$  to be the embeddings  $\mathbb{C}/X \rightarrow \overline{\mathbb{C}}/iX$  and  $\mathbb{C}/Y \rightarrow \overline{\mathbb{C}}/iY$  by Proposition 7.5, with  $L$  and  $\bar{L}$  being the functors pulling back along  $p$  and  $ip$  respectively. The isomorphism  $j \circ L \cong \bar{L} \circ i$  is given by the fact that  $i$  preserves pullbacks, by Proposition 7.4. Moreover,  $\bar{L}$  preserves filtered limits since it is right adjoint to the functor  $\overline{\mathbb{C}}/iY \rightarrow \overline{\mathbb{C}}/iX$  mapping a morphism  $c \rightarrow iY$  to  $c \rightarrow iY \xrightarrow{ip} iX$ . We then get a right adjoint  $\bar{R}$  which ensures that  $ip$  is exponentiable in  $\overline{\mathbb{C}}$ . Moreover, the isomorphism  $i \circ R \cong \bar{R} \circ j$ , where  $R$  is the pushforward along  $p$ , ensures that  $i$  preserves pushforwards along  $p$ .  $\square$

**Corollary 7.8.**  *$\text{Gat}$ , equipped with  $e(p_{\mathcal{G}})$ , is a  $\text{CartExp}$ -category, and  $e: \text{FinGat} \rightarrow \text{Gat}$  is a  $\text{CartExp}$ -functor.*

**PROOF.** By Proposition 7.4,  $\text{Gat}$  has finite limits, preserved by  $e$ . By Corollary 7.7,  $e(p_{\mathcal{G}})$  is exponentiable in  $\text{Gat}$  and  $e$  preserves pushforwards along  $p_{\mathcal{G}}$ .  $\square$

The proof of Theorem 7.1 finally relies on the following observation.

**Proposition 7.9.** *Let  $i: \mathbb{C} \rightarrow \overline{\mathbb{C}}$  be a free filtered completion of a small cartesian category  $\mathbb{C}$  with an exponentiable morphism  $p_{\mathbb{C}}: Y_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ . Consider a  $\text{CartExp}$ -functor  $F: \mathbb{C} \rightarrow \mathbb{D}$  with  $\mathbb{D}$  complete. By*

*universal property of the free filtered completion, we get a functor  $\bar{F}: \overline{\mathbb{C}} \rightarrow \mathbb{D}$  with an isomorphism  $\bar{F} \circ i \cong F$ .*

*Then,  $\bar{F}$  preserves limits and pushforwards along  $ip_{\mathbb{C}}$ .*

**PROOF.** The fact that it preserves limits is a consequence of [1, Proposition 1.45.(ii)]. For preservation of pushforwards, we need to show that the mate of the following natural isomorphism is also an isomorphism.

$$\begin{array}{ccc} \mathbb{D}/X_{\mathbb{D}} & \xrightarrow{p_{\mathbb{D}}^*} & \mathbb{D}/Y_{\mathbb{D}} \\ \uparrow \bar{F} & \cong & \uparrow \bar{F} \\ \overline{\mathbb{C}}/iX & \xrightarrow{ip_{\mathbb{C}}} & \overline{\mathbb{C}}/iY \end{array}$$

By the universal property of the free filtered completion  $\mathbb{C}/Y \rightarrow \overline{\mathbb{C}}/iY$  by Proposition 7.5, it is enough to show that the whiskering of the mate by  $\mathbb{C}/Y \rightarrow \overline{\mathbb{C}}/iY$  is an isomorphism. But this follows from the fact that  $F$  itself preserves pushforwards along  $p$ .  $\square$

We are now ready to prove the claimed bi-initial characterisation of  $\text{Gat}$ .

**PROOF OF THEOREM 7.1.** First,  $(\text{Gat}, e(p_{\mathcal{G}}))$  is a  $\text{CartExp}$ -category by Corollary 7.8. Moreover, it is complete by Proposition 7.4. Consider a  $\text{CartExp}$ -category  $\mathbb{C}$  with all limits. By bi-initiality of  $\text{FinGat}$ , we get a  $\text{CartExp}$ -functor  $F: \text{FinGat} \rightarrow \mathbb{C}$ , which extends to a  $\text{CartExp}$ -functor  $\bar{F}: \text{Gat} \rightarrow \mathbb{C}$  preserving limits by Proposition 7.9.

Now, assume given another  $\text{CartExp}$ -functor  $G: \text{Gat} \rightarrow \mathbb{C}$  preserving limits, and let us show that  $G$  is isomorphic to  $\bar{F}$ , for a unique isomorphism. By the universal property of the free filtered completion  $e: \text{FinGat} \rightarrow \text{Gat}$ , it is enough to show that  $G \circ e$  is isomorphic to  $\bar{F} \circ e$  for a unique morphism, which follows from bi-initiality of  $\text{FinGat}$ , since both are composition of  $\text{CartExp}$ -functors, by Corollary 7.8.  $\square$

## 8 Conclusion and Future Work

We introduced the two-sortification translation of generalised algebraic theories and notably showed that there is a strict coreflection between the categories of models of a GAT and its translation. Our development is heavily based on the bi-initiality of the category of finite GATs [27], as a cartesian category equipped with the exponentiable morphism  $(U: \text{Set}, u: U) \rightarrow (U: \text{Set})$ . We extended this characterisation to account for two-sortification of infinite GATs. Other generalisations remain to be investigated, such as infinitary operations [20], second-order generalised algebraic theories [28], as well as adapting this work to a type-theoretic, possibly univalent, metatheory.

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## A Proofs of the Main Results

In this appendix, we provide some detailed proofs of the main results of the paper.

### A.1 CartExp-structure on FinGat/Fam

The right adjoint to the pullback functor along  $\underline{p}$  can be computed explicitly as follows.

**Proposition A.1.** *In the setting of Proposition 3.5, the right adjoint to the pullback functor along  $\underline{p}$  maps a morphism  $z: Z \rightarrow PX$  to the vertical composite on the left, where the dashed arrow is a morphism in the slice over  $X$  between  $\underline{X} \xrightarrow{\varepsilon} X$  and  $PX \rightarrow X$  obtained as the transpose of the morphism on the right under the adjunction  $p^* \dashv P$ .*

$$\begin{array}{ccc}
 P_2 Z & \xrightarrow{\quad} & PZ \\
 \downarrow & \lrcorner & \downarrow Pz \\
 X_2 & \dashrightarrow & PPX \\
 \downarrow & & \downarrow \\
 PX & & PX
 \end{array}
 \qquad
 \begin{array}{ccc}
 & & Y_2 \\
 & & \downarrow p_2 \\
 & & X_2 \\
 & & \downarrow \\
 & & PX
 \end{array}$$

### A.2 CartExp-structure on $F/\mathbb{C}$

The following proposition is used in the proof of Theorem 2.13.

**Proposition A.2.** *Let  $(\mathbb{D}, p_{\mathbb{D}} : Y \rightarrow X)$  and  $(\mathbb{C}, p_{\mathbb{C}} : Y' \rightarrow X')$  be **CartExp**-categories, and let  $F : \mathbb{D} \rightarrow \mathbb{C}$  be a functor preserving pullbacks along  $p_{\mathbb{D}}$ , and let the following square be a pullback in  $\mathbb{C}$ .*

$$\begin{array}{ccc}
 FY & \xrightarrow{y} & Y' \\
 Fp_{\mathbb{D}} \downarrow & \lrcorner & \downarrow p_{\mathbb{C}} \\
 FX & \xrightarrow{x} & X'
 \end{array}$$

*The comma category  $F/\mathbb{C}$ , whose objects are triples  $(C \in \mathbb{C}, D \in \mathbb{D}, f : FC \rightarrow D)$  and morphisms are commuting squares, can be equipped with a **CartExp**-structure, where the chosen exponentiable morphism is given by the pullback square  $(Fp_{\mathbb{D}}, p_{\mathbb{C}}) : y \rightarrow x$ , and the functor  $F/\mathbb{C} \rightarrow \mathbb{C} \times \mathbb{D}$  is a **CartExp**-functor.*

**PROOF.** The composite

$$(F/\mathbb{C})/x \xrightarrow{(Fp_{\mathbb{D}}, p_{\mathbb{C}})^*} (F/\mathbb{C})/y \xrightarrow{U} F/\mathbb{C}$$

has a right adjoint  $P : F/\mathbb{C} \rightarrow (F/\mathbb{C})/x$  as follows:

$$\begin{array}{ccccc}
 FA & & FPA & \xrightarrow{FP_{\mathbb{D}}A} & FX \\
 \alpha \downarrow & \lrcorner & P\alpha \downarrow & & \downarrow x \\
 B & & P'B & \xrightarrow{P_{\mathbb{C}}B} & X'
 \end{array}$$

where  $P\alpha$  is the transposition of  $F\mathbb{P} \xrightarrow{F(\overline{1_{QA}})} FA \xrightarrow{\alpha} B$  and  $F\mathbb{P}$  is the following pullback.

$$\begin{array}{ccc}
 F\mathbb{P} & \longrightarrow & FPA \\
 \downarrow \lrcorner & & \downarrow FP_{\mathbb{D}}X \\
 FY & \xrightarrow{Fp_{\mathbb{D}}} & FX \\
 y \downarrow \lrcorner & & \downarrow x \\
 Y' & \xrightarrow{p_{\mathbb{C}}} & X'
 \end{array}$$

The upper square is a pullback since  $F$  preserves pullbacks along  $p_D$ . The lower square is the pullback from the assumption.

To prove the adjunction, we use the following characterization of adjoint functors: a functor  $L : \mathbb{C} \rightarrow \mathbb{D}$  has a right adjoint  $R$  if and only if for any object  $d$  of  $\mathbb{D}$ , the comma category  $L/d$  has a terminal object  $(Rd : \mathbb{C}, \alpha : LRd \rightarrow d)$ . Hence, we need to show that for all  $(D \in \mathbb{D}, C \in \mathbb{C}, g : FD \rightarrow C)$  in  $F/\mathbb{C}$ , there exists a morphism  $\theta : LPg \rightarrow g$  in  $F/\mathbb{C}$  such that  $(Pg, \theta)$  is terminal in  $L/g$ . We take  $\theta$  to be the counit  $\epsilon_g$  in  $F/\mathbb{C}$ . To show that  $(Pg, \epsilon_g)$  is terminal in  $L/g$ , we need to show that for all  $\alpha \in (F/\mathbb{C})/x$  and  $\beta : L\alpha \rightarrow g$  in  $F/\mathbb{C}$ , there exists a unique morphism  $\iota : \alpha \rightarrow Pg$  in  $(F/\mathbb{C})/x$ , making the following diagram commute in  $F/\mathbb{C}$ .

$$\begin{array}{ccc} L\alpha & \xrightarrow{L\iota} & LPg \\ & \searrow \beta & \swarrow \epsilon_g \\ & & g \end{array}$$

We take  $\iota$  to be the transposition of  $\beta$ . The above diagram commutes by the definition of  $\beta$ . Such an  $\iota$  is unique by the universal property of the adjunction.

Now to show that the functor  $F/\mathbb{C} \rightarrow \mathbb{C} \times \mathbb{D}$  is a **CartExp**-functor, we need to show that it preserves finite limits, the exponentiable morphism, and the pushforwards along it. Preservation of finite limits is a general fact about comma categories  $F/G$  when  $G$  preserves them. Finally, preservation of the exponentiable morphism and pushforwards along it follow directly from the construction as the right adjoint  $P$  is defined such that it preserves the action of its underlying polynomial functors  $P_{\mathbb{C}}$  and  $P_{\mathbb{D}}$ .  $\square$

**PROOF OF THEOREM 2.13.** Let us start with existence of the natural transformation. The given pullback turns  $F/\mathbb{C}$  into a **CartExp**-category, by Proposition A.2. Thus, we get a bi-initial **CartExp**-functor  $\mathbf{FinGat} \rightarrow F/\mathbb{C}$ . Let  $I : \mathbf{FinGat} \rightarrow \mathbf{FinGat}$  and  $G' : \mathbf{FinGat} \rightarrow \mathbb{C}$  be induced by the composition of **CartExp**-functors  $\mathbf{FinGat} \rightarrow F/\mathbb{C} \rightarrow \mathbf{FinGat} \times \mathbb{C}$ . It follows that  $I$  and  $G'$  are **CartExp**-functors, and are thus isomorphic to the identity endofunctor and  $G$  respectively. Because the projection  $F/\mathbb{C} \rightarrow \mathbf{FinGat} \times \mathbb{C}$  is an isofibration, we get a **CartExp**-functor  $H : \mathbf{FinGat} \rightarrow F/\mathbb{C}$  for which  $G'$  and  $I$  defined as above are exactly  $G$  and the identity functor respectively. The component of the desired natural transformation at  $\Gamma$  is then given by taking the image of  $\Gamma$  by  $H$ .

For uniqueness, consider another natural transformation  $\beta$  between  $F$  and  $G$ , with the same pullback naturality square for  $p_G$ . By universal property of the comma category [7, Proposition 1.6.3], this induces a functor  $\mathbf{FinGat} \rightarrow F/\mathbb{C}$  such that post composition with either projection from  $F/\mathbb{C}$  yields the identity functor or  $G$ . It can be checked that it is a **CartExp**-functor, because  $G$  is. Therefore, it is isomorphic to  $H$ . This isomorphism is easily seen to be an identity, because its whiskering with  $F/\mathbb{C} \rightarrow \mathbf{FinGat} \times \mathbb{C}$  is, by uniqueness of the isomorphism between two bi-initial **CartExp**-functors.  $\square$

### A.3 The Semantic Translation

*Theorem 4.3 (repeated).* The forgetful functor from pointed sets to sets, equipped with the identity natural transformation between the family functors, is exponentiable in  $\mathbf{CAT} // \mathbf{Fam}$ , and the functor  $\mathbf{CAT} // \mathbf{Fam} \rightarrow \mathbf{CAT}$  preserves pushforwards along it. Therefore,

combining with Proposition 4.2,  $\mathbf{CAT} // \mathbf{Fam}$  is a **CartExp**-category and the functor  $\mathbf{CAT} // \mathbf{Fam} \rightarrow \mathbf{CAT}$  is a **CartExp**-functor.

**PROOF.** We start by showing that the following morphism is exponentiable in  $\mathbf{CAT} // \mathbf{Fam}$ :

$$\begin{array}{ccc} \mathbf{PtdSet} & \xrightarrow{p} & \mathbf{Set} \\ & \searrow \text{Id} & \swarrow \delta \\ & & \mathbf{Fam} \end{array}$$

where  $\delta$  maps a set  $X$  to the family defined by  $U = 1$  and  $El = X$ , and  $p$  is the forgetful functor from pointed sets to sets. By Proposition 2.2, we need to show that the composite

$$(\mathbf{CAT} // \mathbf{Fam})/\delta \xrightarrow{p^*} (\mathbf{CAT} // \mathbf{Fam})/(\delta \circ p) \xrightarrow{\text{dom}} \mathbf{CAT} // \mathbf{Fam}$$

has a right adjoint. Similar to the case of  $\mathbf{CAT}$ , explained in Example 2.10, this composite maps an object  $(F : \mathbb{C} \rightarrow \mathbf{Fam}, U : \mathbb{C} \rightarrow \mathbf{Set}, \gamma : \delta \circ U \Rightarrow F)$  in  $(\mathbf{CAT} // \mathbf{Fam})/\delta$  to the object  $\int U \rightarrow \mathbb{C} \xrightarrow{F} \mathbf{Fam}$  in  $\mathbf{CAT} // \mathbf{Fam}$ , where  $\int U$  is the category of elements of  $U$ , and the morphism  $\int U \rightarrow \mathbb{C}$  is the associated discrete opfibration.

Before defining the right adjoint, we first note that a morphism of the form

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{H} & \mathbb{D} \\ & \searrow F & \swarrow G \\ & & \mathbf{Fam} \end{array}$$

in  $\mathbf{CAT} // \mathbf{Fam}$  can be equivalently described as the following commuting square, using the equivalence between  $\mathbf{Fam}$  and  $\mathbf{Set}^{\rightarrow}$ :

$$\begin{array}{ccc} G_1 Hc & \xrightarrow{\alpha_1 c} & F_1 c \\ \downarrow & & \downarrow \\ G_0 Hc & \xrightarrow{\alpha_0 c} & F_0 c. \end{array}$$

The claimed right adjoint maps an object  $\mathbb{D} \xrightarrow{G} \mathbf{Fam}$  in  $\mathbf{CAT} // \mathbf{Fam}$  to an object  $(F' : P\mathbb{D} \rightarrow \mathbf{Fam}, U' : P\mathbb{D} \rightarrow \mathbf{Set}, \gamma' : \delta \circ U' \Rightarrow F')$  in  $(\mathbf{CAT} // \mathbf{Fam})/\delta$ , where  $P$  is the right adjoint to the pullback functor for the case of  $\mathbf{CAT}$  defined in Example 2.10,  $U'$  is defined as the composite  $P\mathbb{D} \xrightarrow{P(H)} P\mathbf{Fam} \rightarrow \mathbf{Set}$ ,  $F'$  maps an object

$$(X \in \mathbf{Set}, f : X \rightarrow \mathbb{D}) \text{ to the family identified by } \begin{array}{c} X + \sum_x G_1 f(x) \\ \downarrow \\ 1 + \sum_x G_0 f(x) \end{array},$$

and  $\gamma'$  is identified by the following commuting square:

$$\begin{array}{ccc} X & \xrightarrow{f} & X + \sum_x G_1 f(x) \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{f_!} & 1 + \sum_x G_0 f(x). \end{array}$$

To prove the adjunction, it suffices to construct the homset bijection and show that it is natural in  $X$ . Naturality in  $X$  then implies that there exists a unique way to extend the map on objects to a functor which is natural in  $Y$ .

For the homset bijection, let  $(F : \mathbb{C} \rightarrow \mathbf{Fam}, U : \mathbb{C} \rightarrow \mathbf{Set}, \gamma : \delta \circ U \Rightarrow F)$  be an object in  $(\mathbf{CAT} // \mathbf{Fam})/\delta$  and  $\mathbb{D} \xrightarrow{G} \mathbf{Fam}$  be an

object in  $\text{CAT} // \mathbf{Fam}$ . We need to show a bijection between the homsets identified by the following two commuting square:

$$\begin{array}{ccc} G_1H(c, u) & \xrightarrow{\alpha_1(c, u)} & F_1c \\ \downarrow & & \downarrow \\ G_0H(c, u) & \xrightarrow{\alpha_0(c, u)} & F_0c \end{array}, \quad \begin{array}{ccc} U_c + \sum_u G_1H(c, u) & \xrightarrow{\beta_1} & F_1c \\ \downarrow & & \downarrow \\ 1 + \sum_u G_0H(c, u) & \xrightarrow{\beta_0} & F_0c \end{array}$$

such that  $\beta$  precomposition by  $\gamma'$  is equal to  $\gamma$ :

$$\begin{array}{ccc} U_c & \xrightarrow{\gamma_1} & U_c + \sum_u G_1H(c, u) \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\gamma_0} & 1 + \sum_u G_0H(c, u) \end{array} \quad \begin{array}{ccc} & \xrightarrow{\beta_1} & F_1c \\ & & \downarrow \\ & \xrightarrow{\beta_0} & F_0c \end{array} = \begin{array}{ccc} U_c & \xrightarrow{\gamma_1 c} & F_1c \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\gamma_0 c} & F_0c \end{array}$$

Note that the equation  $\beta \circ \gamma' = \gamma$  implies that  $\beta_1 \circ \iota_1 = \gamma_1$  and  $\beta_0 \circ \iota_0 = \gamma_0$ . Therefore, the data of  $\beta$  is equivalent to that of the following commuting square:

$$\begin{array}{ccc} \sum_u G_1H(c, u) & \xrightarrow{\beta_{r_1}} & F_1c \\ \downarrow & & \downarrow \\ \sum_u G_0H(c, u) & \xrightarrow{\beta_{r_0}} & F_0c \end{array}$$

Finally, by the universal property of the coproduct  $\sum_u$ , specifying  $\beta_{r_i} : \sum_u G_iH(c, u) \rightarrow F_ic$  is equivalent to specifying for each  $u \in U_c$ , a morphism  $G_iH(c, u) \rightarrow F_ic$ . These families of morphisms are exactly the data of the natural transformation  $\alpha$ .

The naturality of  $\alpha$  follows from the naturality of  $\beta$  and the construction above. In particular,  $\beta$  is natural in  $c$ , and by restricting this naturality square along the coproduct injections, we obtain the naturality of  $\alpha$ .

It remains to show the naturality of this bijection in  $X$ , that is for every morphism  $f : X \rightarrow X'$  in  $(\text{CAT} // \mathbf{Fam})/\delta$  and every object  $Y \in \text{CAT} // \mathbf{Fam}$ , the following square commutes:

$$\begin{array}{ccc} \text{Hom}_{\text{CAT} // \mathbf{Fam}}(LX', Y) & \xrightarrow{\Psi_{X', Y}} & \text{Hom}_{(\text{CAT} // \mathbf{Fam})/\delta}(X', RY) \\ (-) \circ L(f) \downarrow & & \downarrow (-) \circ f \\ \text{Hom}_{\text{CAT} // \mathbf{Fam}}(LX, Y) & \xrightarrow{\Psi_{X, Y}} & \text{Hom}_{(\text{CAT} // \mathbf{Fam})/\delta}(X, RY), \end{array}$$

where  $L$  is the composition of the pullback along  $p$  and  $\text{dom}$ , and  $R$  is  $P_{\text{CAT} // \mathbf{Fam}}$ , the right adjoint defined above. The bijection  $\Psi$  acts by taking coproducts indexed by the elements of  $U_c$  and copairing with  $\gamma$ , both of which commute with reindexing of the indexing sets  $U_c$  induced by  $f$ . Hence, the above square commutes.

This concludes proving the first part of the theorem. Showing that the forgetful functor  $\text{CAT} // \mathbf{Fam} \rightarrow \text{CAT}$  preserves pushforwards along the exponentiable morphism is straightforward, as  $P_{\text{CAT} // \mathbf{Fam}}$  is  $P_{\text{CAT}}$  on the level of the underlying categories, by definition.  $\square$

*Proposition 4.10 (repeated).*  $\text{CAT} //_{\mathbf{c}} \mathbf{Fam}$  is a sub- $\text{CartExp}$ -category of  $\text{CAT} // \mathbf{Fam}$ , in the sense that it has finite limits, the exponentiable morphism from Theorem 4.3 is in  $\text{CAT} //_{\mathbf{c}} \mathbf{Fam}$  and is again exponentiable, and finally, the functor  $\text{CAT} //_{\mathbf{c}} \mathbf{Fam} \rightarrow \text{CAT} // \mathbf{Fam}$  preserves finite limits and pushforwards along this exponentiable morphism.

**PROOF.** It is straightforward to check that the terminal object in  $\text{CAT} // \mathbf{Fam}$  is also in  $\text{CAT} //_{\mathbf{c}} \mathbf{Fam}$ . As for pullbacks, they can be computed in  $\text{CAT} // \mathbf{Fam}$  as follows: first compute the pullback of the categories in  $\text{CAT}$ . The functor from this pullback to  $\mathbf{Fam}$  is given by the pushout of the whiskering of the given natural transformations by the pullback projections. If these natural transformations are cartesian, we can compute this pushout so that the pushouts coprojections are also cartesian. This results in a pullback in  $\text{CAT} //_{\mathbf{c}} \mathbf{Fam}$  which is also a pullback in  $\text{CAT} // \mathbf{Fam}$ . It follows that  $\text{CAT} //_{\mathbf{c}} \mathbf{Fam}$  has finite limits and that the inclusion  $\text{CAT} //_{\mathbf{c}} \mathbf{Fam} \rightarrow \text{CAT} // \mathbf{Fam}$  preserves them.

The proof of Theorem 4.3 can directly be extended to the case of  $\text{CAT} //_{\mathbf{c}} \mathbf{Fam}$  to show that the chosen exponentiable morphism of  $\text{CAT} // \mathbf{Fam}$  is also exponentiable in  $\text{CAT} //_{\mathbf{c}} \mathbf{Fam}$ . It is then straightforward that the inclusion preserves pushforwards along this exponentiable morphism.  $\square$

*Proposition 4.11 (repeated).* The mapping  $(\mathbb{C} \xrightarrow{F} \mathbf{Fam}) \mapsto (F / {}_{\mathbf{c}}\mathbf{Fam} \rightarrow {}_{\mathbf{c}}\mathbf{Fam})$  extends to a  $\text{CartExp}$ -functor from  $\text{CAT} //_{\mathbf{c}} \mathbf{Fam}$  to  $\text{CAT} / \mathbf{Fam}$ .

**PROOF.** The induced functor from  $\text{CAT} //_{\mathbf{c}} \mathbf{Fam}$  to  $\text{CAT} / \mathbf{Fam}$  can be characterised as the right adjoint of the obvious embedding of  $\text{CAT} / \mathbf{Fam}$  into  $\text{CAT} //_{\mathbf{c}} \mathbf{Fam}$ .

We abstract the situation as follows: we have an adjunction  $L \dashv R : \mathbb{C} \rightarrow \mathbb{D}$  between categories such that

- $(\mathbb{C}, X_{\mathbb{C}} \xrightarrow{p_{\mathbb{C}}} Y_{\mathbb{C}})$  and  $(\mathbb{D}, X_{\mathbb{D}} \xrightarrow{p_{\mathbb{D}}} Y_{\mathbb{D}})$  are  $\text{CartExp}$ -categories;
- $Rp_{\mathbb{C}} \cong p_{\mathbb{D}}$ ;
- the left adjoint  $L$  preserves pullbacks;
- the naturality square of the counit for the exponentiable morphism  $p_{\mathbb{C}}$  is a pullback.

We can show that the right adjoint  $R$  preserves pushforwards along  $p_{\mathbb{C}}$  and is thus a  $\text{CartExp}$ -functor as desired. Consider indeed the square below commuting up to isomorphism, where the vertical functors are induced by  $R$  and the isomorphism  $Rp_{\mathbb{C}} \cong p_{\mathbb{D}}$ .

$$\begin{array}{ccc} \mathbb{C} / X_{\mathbb{C}} & \xrightarrow{p_{\mathbb{C}}^*} & \mathbb{C} / Y_{\mathbb{C}} \\ \downarrow & & \downarrow \\ \mathbb{D} / X_{\mathbb{D}} & \xrightarrow{p_{\mathbb{D}}^*} & \mathbb{D} / Y_{\mathbb{D}} \end{array}$$

Preserving pushforwards along  $p_{\mathbb{C}}$  requires the mate natural transformation  $P_{\mathbb{C}}R \rightarrow Rp_{\mathbb{D}}$  to be an isomorphism. Note that we have another mate natural transformation, induced by the left adjoints of the vertical functors. It is easy to check that this one is an isomorphism. The result follows from the fact that in this situation, one mate natural transformation is an isomorphism if and only if the other one is.  $\square$

#### A.4 Two-Sortification is Fully Faithful

In this section, we prove a generalisation of the key lemma stated in the proof of Proposition 6.11.

**Notation A.3.** Given a morphism  $f : \Omega \rightarrow PX$ , we denote the composition  $\Omega \xrightarrow{f} PX \rightarrow X_{\mathcal{G}}$  by  $\cup_f : \Omega \rightarrow X_{\mathcal{G}}$ .

Given  $f: \Omega \rightarrow PX$  and  $A \in \text{Tm}(U_f)$ , we denote the composition  $\Omega \rightarrow TX_{\mathcal{G}} \xrightarrow{\varepsilon_{X_{\mathcal{G}}}} X_{\mathcal{G}}$  by  $\text{El}_f A$ , where the first morphism is the one corresponding to  $A$  and  $f$  by the universal property of the pullback defining  $TX_{\mathcal{G}}$ , and the second morphism  $\varepsilon_{X_{\mathcal{G}}}$  is the counit at  $X_{\mathcal{G}}$  of the adjunction  $p_{\mathcal{G}}^* \dashv P$ .

**Remark A.4.** The notations are motivated by the fact that a morphism  $f: \Omega \rightarrow \mathbf{Fam}$  corresponds to a pair of sorts  $\Omega \vdash U_f: \mathbf{Set}$  and  $\Omega \vdash \text{El}_f: U_f \rightarrow \mathbf{Set}$ , an element of  $\text{Tm}(U_f)$  corresponds to a term  $\Omega \vdash A: U_f$ , and an element of  $\text{Tm}(\text{El}_f A)$  corresponds to a term  $\Omega \vdash t: \text{El}_f A$ .

From the following proposition, we easily recover the key lemma stated in the proof of Proposition 6.11, by considering the initial model of  $T\Omega$  and  $f$  to be the projection  $T\Omega \rightarrow \mathbf{Fam}$ .

**Proposition A.5.** *Let  $0_{\Omega}$  denote the initial model of  $\Omega$  from Proposition 5.3. For any morphism  $f: \Omega \rightarrow \mathbf{Fam}$ , the family  $\llbracket f \rrbracket(0_{\Omega})$  is  $(\text{Tm}(\text{El}_f A))_{A \in \text{Tm}(U_f)}$ .*

The proof relies on the following lemma.

**Lemma A.6.** *Let  $\alpha$  be a natural transformation as induced by Theorem 2.13, i.e., any natural transformation between two functors  $F, G: \mathbf{FinGat} \rightarrow \mathbb{C}$ , such that  $G$  is a strict  $\mathbf{CartExp}$ -functor,  $F$  preserves pullbacks along  $p_{\mathcal{G}}$ , and the naturality square of  $\alpha$  at  $p_{\mathcal{G}}$  is a pullback square in  $\mathbb{C}$ . Then, for any theory  $\Gamma$ , the morphism  $F\Gamma \xrightarrow{\alpha_{\Gamma}} G\Gamma \cong P_{\mathbb{C}}G\Gamma$  is the composition below left, where  $\beta_{\Gamma}: F\Gamma \rightarrow P_{\mathbb{C}}F\Gamma$  is the transpose of the dashed morphism below*

*right with respect to the adjunction  $p_{\mathbb{C}}^* \dashv P_{\mathbb{C}}$ , with  $\varepsilon_{\Gamma}: \Gamma' \rightarrow \Gamma$  denoting the counit component at  $\Gamma$ .*

$$\begin{array}{ccc}
 F\Gamma & & \\
 \downarrow \beta_{\Gamma} & & \Gamma' \rightarrow FY_{\mathcal{G}} \\
 P_{\mathbb{C}}F\Gamma & & \downarrow \downarrow Fp_{\mathcal{G}} \\
 \downarrow P_{\mathbb{C}}\alpha_{\Gamma} & \swarrow F\varepsilon_{\Gamma} & F\Gamma' \rightarrow FX_{\mathcal{G}} \\
 P_{\mathbb{C}}G\Gamma & & \downarrow \\
 \downarrow \cong & & F\Gamma \\
 G\Gamma & & 
 \end{array}$$

PROOF. This morphism and  $\alpha_{P\Gamma}$  have the same transpose.  $\square$

PROOF OF PROPOSITION A.5. We consider the natural transformation  $\alpha_{\Gamma}: \text{hom}(\Omega, \Gamma) \rightarrow \llbracket \Gamma \rrbracket$  mapping  $\sigma$  to  $\llbracket \sigma \rrbracket(0_{\Omega})$ ; the naturality square at  $p_{\mathcal{G}}$  is the pullback (2). Note that  $\llbracket f \rrbracket(0_{\Omega})$  is  $\alpha_{PX_{\mathcal{G}}}(f)$ . By Lemma A.6, we conclude that  $\alpha_{PX_{\mathcal{G}}}$  is the following map

$$\text{hom}(\Omega, PX_{\mathcal{G}}) \xrightarrow{\beta_{\Gamma}} \text{Fam}(\text{hom}(\Omega, X_{\mathcal{G}})) \xrightarrow{\text{Fam}(\alpha_{X_{\mathcal{G}}})} \text{Fam}(\mathbf{Set}) \quad (3)$$

Note that  $\alpha_{X_{\mathcal{G}}}$  maps  $A: \Omega \rightarrow X_{\mathcal{G}}$  to  $\llbracket A \rrbracket 0_{\Omega} = \text{Tm}(A)$ . It remains to compute  $\beta_{\Gamma}: \text{hom}(\Omega, PX_{\mathcal{G}}) \rightarrow \text{Fam}(\text{hom}(\Omega, X_{\mathcal{G}}))$ . As the transpose of  $\varepsilon_{X_{\mathcal{G}}} \circ -: \text{hom}(\Omega, TX_{\mathcal{G}}) \rightarrow \text{hom}(\Omega, X_{\mathcal{G}})$ , the function  $\beta_{\Gamma}$  maps  $f: \Omega \rightarrow PX_{\mathcal{G}}$  to the family  $(\text{El}_f A)_{A \in \text{Tm}(U_f)}$ . It is now clear that the map (3) yields the expected family when applied to  $f$ .  $\square$